PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Large-Scale Structure of Homogeneous Turbulence

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Phil. Trans. R. Soc. Lond. A 1956 **248**, 369-405
doi: 10.1098/rsta.1956.0002

THE LARGE-SCALE STRUCTURE OF HOMOGENEOUS TURBULENCE

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(*Communicated by Sir Geoffrey Taylor, F.R.S.—Received* 22 *March* 1955)

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The starting-point for this paper lies in some results obtained by Proudman & Reid (1954) for isotropic turbulence with zero fourth-order cumulants. They showed in that case that the quantity $\overline{u^2}\int_0^\infty r^4 f(r)\,\mathrm{d}r$ is not a dynamical invariant and that the first time derivative of the triple correlation $k(r)$ is of order r^{-4} when r is large. The customary assumption that all velocity cumulants in homogeneous turbulence are exponentially small for large separations, and the consequent results, about the large-scale structure of the motion and about the final period of decay of the turbulence, are thus suspect, and we have redeveloped the whole subject *ab initio*.

The fallacy in the old assumption of exponentially small cumulants can be ascribed to the action of pressure forces, which are local in their effect but which have values determined instantaneously by the whole velocity field. If at some initial instant a finite region only of an infinite fluid is in motion, at subsequent instants pressure forces generate a surrounding irrotational velocity distribution that falls off as some integral power of the distance from the central region. Likewise, the action of pressure forces in homogeneous turbulence is to ensure the development of algebraic asymptotic forms of velocity cumulants, the analogue of the finite region of initial motion being a volume of the fluid over which the vorticity is effectively correlated. However, an essential difference between the two cases is that in homogeneous turbulence pressure forces also build up long-range statistical connexions in the vorticity distribution.

Having recognized why the old assumption is wrong, it is necessary to consider what kinds of asymptotic forms of velocity cumulants (for large separation) are dynamically persistent, and to consider in particular what asymptotic forms are likely to occur when homogeneous turbulence is generated in the usual way by setting a regular array of rods across a uniform stream. We have been able to find only one kind of large-scale structure that is unchanged by dynamical action, and this is also the kind of large-scale structure that develops from a plausibly idealized initial condition representing the effect of the grid on the stream. This initial condition, according to the hypothesis on which the positive results of this paper are based, is that there is a virtual origin in time at which all integral moments of cumulants of the velocity field converge. The important consequence of this hypothesis is that the effect of pressure forces is subsequently to develop asymptotic forms that are integral power-laws.

It is shown, from a consideration of all the time derivatives at the initial instant, that the velocity covariance $\overline{u_i u_j'}$ in general becomes of order r^{-5} when the separation r is large, the leading term having the property that it makes no contribution to the vorticity covariance $\overline{\omega_i \omega_j'}$, which becomes of order r^{-8}. This semi-irrotational property of the asymptotic form, which arises from the fact that pressure forces act only indirectly on the vorticity, allows the asymptotic form of $\overline{u_i u_j'}$ to be determined explicitly. By methods that are new in turbulence theory and that involve a good deal of tensor manipulation, it is found that

$$\overline{u_i u_j'} = \tfrac{1}{4} C_{pqmn}\left(\delta_{ip}\nabla^2 - \frac{\partial^2}{\partial r_i \partial r_p}\right)\left(\delta_{jq}\nabla^2 - \frac{\partial^2}{\partial r_j \partial r_q}\right)\frac{\partial^2 r}{\partial r_m \partial r_n} + O(r^{-6}),$$

when r is large, where the coefficient C_{pqmn} is related to the fourth integral moment of $\overline{\omega_i \omega_j'}$ in a known way. There is a corresponding expression for the leading term in the spectrum tensor at small wave-numbers, which is now not analytic. The spectrum function giving the distribution of energy with respect to wave-number magnitude k is in general of the form

$$E(k) = Ck^4 + O(k^5 \ln k),$$

when k is small.

Corresponding expressions are found for the asymptotic forms of the various terms occurring in the dynamical equation giving the rate of change of $\overline{u_i u_j'}$. Both the inertia and pressure terms in this equation are found to be of order r^{-5}, and as a consequence the coefficient C_{pqmn} (and likewise C) is not a dynamical invariant. It is shown that the integral $\int \overline{u_i u_j'} r_m r_n \,\mathrm{d}\mathbf{r}$ (which exists, despite the apparent logarithmic divergence at large r) is uniquely related to C_{pqmn}, and it too varies during the decay, contrary to past belief.

The final period of decay is examined afresh, and it is found that the energy then varies as $(t-t_0)^{-\frac{5}{2}}$, which is also the result found experimentally; the power $(-\frac{5}{2})$ arises from the fact that the spectrum tensor is of order k^2 when the wave-number k is small, and is unaffected by the non-analytic character of that leading term. The covariance $\overline{u_i u_j'}$ does not have a simple form in the final period; it is determined by the parameter $\int \overline{u_i u_j'} r_m r_n \,\mathrm{d}\mathbf{r}$ alone, and this parameter depends on the previous history of the decay in a complicated way. It is rather puzzling that measurements indicate that the longitudinal correlation coefficient has a simple Gaussian form in the final period of decay, as would be the case for an analytic spectrum. We suggest this observation may be true only for turbulence of very low initial Reynolds number, for which the non-analytic part of the spectrum tensor has little time to develop.

Finally, the results are specialized to correspond to turbulence which is completely isotropic. For reasons related to the symmetry, $\overline{u_i u_j'}$ is now no larger than $O(r^{-6})$ when r is large (we have been unable to determine the exact order), and the leading term in the spectrum tensor, of order k^2, is analytic. As suggested by Proudman & Reid's work, the triple correlation $k(r)$ is of order r^{-4} when r is large and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\overline{u^2}\int_0^\infty r^4 f(r)\,\mathrm{d}r\right\} = (\overline{u^2})^{\frac{3}{2}} \lim_{r\to\infty} r^4 k(r).$$

1. The background to the investigation

An aspect of homogeneous turbulence which has attracted much attention in theoretical research in the subject is the motion of those components that are associated with very large length-scales. This attraction has been due largely to the apparent simplicity of the problem and to the possibility of making the dynamical theory unusually complete. Nevertheless, the dynamical results have not been without interest in themselves. For, according to the theory, although the large-scale components play little part in the general mechanics of the turbulence when the Reynolds number is high, they dominate the motion in the last stages of decay when the turbulent energy has been reduced to a low level by viscous dissipation. The theory has predicted in some detail the properties of the turbulence in this final period of decay, and it has been possible to compare the results of experiments on homogeneous turbulence with predictions whose basis is entirely theoretical. The satisfactory outcome of this comparison is now common knowledge.

Since, however, we shall be concerned in this paper with a re-examination of the theoretical ideas, it will be useful to recall the developments mentioned above in somewhat greater detail. A unified treatment of the relevant theory has been given by Batchelor (1953), and it will suffice here to quote the main results in a form suited to our subsequent analysis. These concern the velocity covariance

$$R_{ij}(\mathbf{r}) = \overline{u_i(\mathbf{x})\, u_j(\mathbf{x}+\mathbf{r})},$$

where $\mathbf{u}(\mathbf{x})$ is the velocity at the point $\mathbf{x}$, and the tensors which govern its dynamical behaviour, namely

$$S_{ijk}(\mathbf{r}) = \overline{u_i(\mathbf{x})\, u_j(\mathbf{x})\, u_k(\mathbf{x}+\mathbf{r})}$$

and

$$P_i(\mathbf{r}) = \overline{p(\mathbf{x})\, u_i(\mathbf{x}+\mathbf{r})},$$

where $p(\mathbf{x})$ is the ratio of the pressure at the point $\mathbf{x}$ to the uniform density of the fluid.

Now, the kinematical assumption on which the theory has hitherto been based is that integral moments of each of the above tensors, such as

$$\int r_m r_n \ldots (N \text{ factors})\; R_{ij}(\mathbf{r})\, \mathrm{d}\mathbf{r},$$

where the integration is over all values of $\mathbf{r}$, converge for all values of N occurring in the analysis. Actually, the assumption has been made only for values of N up to, and including, three, but the central idea has always been that the relevant tensors are exponentially small at large values of $r = |\mathbf{r}|$; otherwise, there would be little *a priori* justification for assuming the existence of particular integral moments. Since, at large values of r, the tensors like $R_{ij}(\mathbf{r})$ measure the statistical correlation between conditions at widely separated points in the turbulence, and since the devices ordinarily used to generate homogeneous turbulence have a structure that is periodic, with finite wavelength, it has usually been believed that there is no great loss of generality in making such an assumption.* It then follows from the equations governing the homogeneous turbulent motion of an incompressible fluid that

$$\int R_{ij}(\mathbf{r})\, \mathrm{d}\mathbf{r} = 0 = \int r_m R_{ij}(\mathbf{r})\, \mathrm{d}\mathbf{r}, \tag{1·1}$$

and

$$\frac{\partial}{\partial t}\int r_m r_n R_{ij}(\mathbf{r})\, \mathrm{d}\mathbf{r} = 0. \tag{1·2}$$

It is customary to express these results in terms of the spectrum tensor

$$\Phi_{ij}(\mathbf{k}) = \frac{1}{8\pi^3}\int R_{ij}(\mathbf{r})\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\, \mathrm{d}\mathbf{r}. \tag{1·3}$$

* Birkhoff (1954) has recently recorded the opinion that convergence of integral moments of $R_{ij}(\mathbf{r})$ cannot generally be expected.

Thus, for small values of $k = |\mathbf{k}|$,

$$\Phi_{ij}(\mathbf{k}) = -\frac{1}{4\pi^2} L_{ijmn} k_m k_n + O(k^3), \tag{1·4}$$

where

$$L_{ijmn} = \frac{1}{4\pi} \int r_m r_n R_{ij}(\mathbf{r}) \, \mathrm{d}\mathbf{r}, \tag{1·5}$$

and the first few terms of the Taylor expansion are justified by the existence of the corresponding integral moments of $R_{ij}(\mathbf{r})$. The way in which the second-order integral moment of $R_{ij}(\mathbf{r})$ describes the motion of the largest-scale components is then more clear, and the result (1·2) states that the asymptotic form of the spectrum function at small k is constant during the entire life history of the turbulence. In the case of isotropic turbulence, the result (1·2) is due to Loitsiansky (1939), and the corresponding spectrum result to Lin (1947). The general results for homogeneous turbulence are due to Batchelor (1949).

The further predictions of the theory are all concerned with conditions in the final period of decay when the intensity of turbulence is sufficiently small to permit the total neglect of inertia and pressure forces. The dynamical equation for $\Phi_{ij}(\mathbf{k})$ is then

$$\frac{\partial \Phi_{ij}(\mathbf{k})}{\partial t} = -2\nu k^2 \Phi_{ij}(\mathbf{k}), \tag{1·6}$$

and the assumption that the first three integral moments of $R_{ij}(\mathbf{r})$ exist permits the derivation of the complete asymptotic solutions for the velocity covariance and the spectrum tensor. Thus, one finds (see Batchelor 1953) that, as $t - t_0 \to \infty$,

$$R_{ij}(\mathbf{r}) \sim -\frac{L_{ijmn}}{(2\pi)^{\frac{1}{2}} \lambda^5} \left[\delta_{mn} - \frac{r_m r_n}{\lambda^2} \right] \mathrm{e}^{-r^2/2\lambda^2}, \tag{1·7}$$

where

$$\lambda^2 = 4\nu(t - t_0),$$

and the constant of integration t_0 is the virtual origin in time of the final period of decay. This is the prediction that compares so favourably with the results of experiments, both in the distribution of the covariance as a function of $\mathbf{r}$, and in the temporal decay of the energy tensor $\overline{u_i(\mathbf{x})\, u_j(\mathbf{x})}$. The assumption that the integral moments of $S_{ijk}(\mathbf{r})$ and $P_i(\mathbf{r})$ converge, is not used in deriving (1·7), but is needed for the more powerful result (1·2) concerning the earlier stages of decay, on which the experimental evidence cited provides no check.

However, indirect support for the result (1·2) follows from a more refined argument about conditions in the final period of decay. For, according to (1·2), the tensor coefficient in (1·7) is the same as that which describes the motion of the largest eddies in the early stages of decay. It follows that a field of homogeneous turbulence in which, initially, the largest eddies are anisotropic, but which is otherwise substantially isotropic, will ultimately become markedly anisotropic when the largest eddies are all that remain. Moreover, anisotropy in the largest eddies is normally to be expected when the mechanism for generating the turbulence has any directional properties, even though the mechanical processes might distribute the bulk of the energy isotropically. In the case of wind-tunnel turbulence, therefore, where the customary bi-plane grid may be regarded as an axi-symmetric generator of approximately isotropic turbulence, the fact that Batchelor & Stewart (1950) have observed the above tendency to anisotropy provides a partial verification of the prediction (1·2).

However, it must be stressed that no direct measurements of the decay of L_{ijmn} have yet been made. Indeed, from the many measurements of the velocity covariance that have been made, there is no real indication that the integral in question converges (i.e. even in a practical numerical sense; the computations of Stewart & Townsend (1951), for instance, suggest that at least 40 % of the contributions to the integral arise from values of r greater than those for which measurements have been made). Nevertheless, the satisfactory nature of the remaining experimental evidence and the apparent generality of the theoretical assumptions have together been responsible for the common feeling that the analysis of the large-scale components represents one of the most surely established parts of the theory of homogeneous turbulence.

The recent work of Proudman & Reid (1954), however, has brought to light a serious flaw in the theory. For isotropic turbulence, they found that, at an instant at which the integral moments of $R_{ij}(\mathbf{r})$ and $S_{ijk}(\mathbf{r})$ converge, the rate of change of $S_{ijk}(\mathbf{r})$ is in general of order r^{-4} for large values of r. It follows that perhaps the first, and certainly the second and all higher, order integral moments of $S_{ijk}(\mathbf{r})$ diverge at subsequent instants, and that some of the assumptions underlying the theory described above are invalid. Actually, Proudman & Reid only show rigorously that the coefficient of r^{-4} is non-zero for the special case in which fourth-order cumulants of the joint probability distribution of the velocity vector at three points in space vanish, but they present plausible arguments for the validity of the result in the general case of an arbitrary probability distribution. In any case, since the basic assumptions of the earlier work were strictly kinematical, the invalidity of these assumptions for a kinematically possible probability distribution like one with zero fourth-order cumulants is itself sufficiently disturbing to warrant further investigation.

Proudman & Reid go on to point out some of the changes in the theory that are necessitated by the new-found behaviour of $S_{ijk}(\mathbf{r})$ at large values of r. In isotropic turbulence, for instance, it appears that, if the integral moments of $R_{ij}(\mathbf{r})$ are still assumed to exist (although there is now less justification for such an assumption), no further inconsistencies arise provided the integral L_{ijmn} is allowed to have a finite rate of change during decay. More precisely, if $\overline{u^2}f(r)$ and $(\overline{u^2})^{\frac{3}{2}}k(r)$ are the double and triple velocity correlation functions usually denoted by these symbols, the scalar equation defining the rate of change of the isotropic tensor L_{ijmn} is

$$\frac{\partial}{\partial t}\int_0^\infty \overline{u^2}r^4f(r)\,\mathrm{d}r = (\overline{u^2})^{\frac{3}{2}}\lim_{r\to\infty} r^4k(r). \tag{1·8}$$

The right-hand side of (1·8) arises entirely from inertia forces, and since these are very small in the final period of decay the only significant change represented by (1·8) concerns the decay of L_{ijmn} during the earlier stages when the Reynolds number is not small. As has already been pointed out, this is an aspect of the theory that has never been verified or disproved experimentally.

The general case of homogeneous turbulence is also considered briefly by Proudman & Reid, who suggest that, in this case, an analytic expansion of the form (1·4) is unlikely to be valid for all t, for dynamical reasons. In other words, they cast doubt on the convergence of the integral L_{ijmn}. In actual fact, of course, the discovery that $S_{ijk}(\mathbf{r}) = O(r^{-4})$ for large values of r immediately suggests that statistical connexions between *all* quantities at widely separated points in homogeneous (including isotropic) turbulence are much greater than has hitherto been supposed. If this is the case, then the theoretical basis of many of the predictions that apparently have been confirmed by experiments is removed. We therefore felt that an entirely new examination of the problem was necessary and that all previous assumptions and theoretical results should be put into the melting-pot. Thus, our aim in this paper is to reconsider the large-scale structure of homogeneous turbulence in general, and in particular to account for the experimental data about turbulence generated by a grid of rods in a wind tunnel.

2. The nature of the problem

Proudman & Reid's work suggests, and analysis presented in later sections shows definitely, that the important point overlooked in previous investigations is the severe restriction imposed on the large-scale kinematical structure of homogeneous turbulence by the dynamical processes taking place. Proudman & Reid encounter this restriction only in so far as it affects the tensor $S_{ijk}(\mathbf{r})$, and we now have to examine its effect on all mean values required in the analytical description of the large-scale motion. For the time being, the problem may be regarded as consisting of two parts, namely, that of finding general large-scale kinematical structures which can persist in time (through being unchanged by dynamical processes), and that of finding the particular structure that is relevant to wind-tunnel turbulence. Actually, we have been able to find a solution of the former problem only by making a direct appeal to practical methods of generating turbulence in a wind-tunnel (as described in §§3 and 4), but it is possible, and useful, to consider in isolation several aspects of the general problem, and this we do in the present section.

Some indication of the nature of the dynamical restriction on the large-scale structure of homogeneous turbulence may be obtained by considering the mechanics of a simpler type of flow of an unbounded fluid in which $|\mathbf{u}(\mathbf{x})| \to 0$ as $x = |\mathbf{x}| \to \infty$. Such a motion is not strictly relevant to the problem of homogeneous turbulence, in which the velocity and all its derivatives extend throughout all space in a statistically uniform manner. But there is, nevertheless, a clear (though not necessarily close) analogy between the volume over which a quantity is correlated with itself in homogeneous turbulence, and the region over which this quantity is appreciably different from zero in the type of flow defined above. In this way, the above flow may be regarded as corresponding to the motion of a single 'eddy' in homogeneous turbulence, and an examination of its properties at large values of x may be expected to yield some information concerning the large-scale structure of homogeneous turbulence.

With this analogy in mind, the aspect of the flow in which we are primarily interested is the restriction that the dynamical processes impose on the orders of magnitude of the velocity and its derivatives at large values of x. In particular, we are interested in the dynamical significance of the difference between flows in which these quantities are exponentially and algebraically small. For, the difference between the corresponding forms of the correlation tensors at large values of r in homogeneous turbulence represents the essential difference between the assumptions of the older theory and the new suggestions mentioned in §1.

We consider first the distribution of acceleration in a flow at an instant at which the distribution of velocity is such that $|\mathbf{u}(\mathbf{x})| \to 0$ exponentially rapidly as $x \to \infty$. In the absence of external forces, the forces producing this acceleration are the viscous stresses and pressure gradients. The viscous stresses at any point depend only on the local velocity distribution and are therefore exponentially small at large values of x. The pressure distribution in the neighbourhood of any point, on the other hand, is built up of contributions that are communicated to that neighbourhood from all parts of the fluid. Thus, in an unbounded fluid, the pressure is given by

$$p(\mathbf{x}) = \frac{1}{4\pi}\int \frac{\partial^2 u_i(\mathbf{x}')u_j(\mathbf{x}')}{\partial x_i' \partial x_j'} \frac{d\mathbf{x}'}{|\mathbf{x}'-\mathbf{x}|}, \tag{2·1}$$

which depends upon the entire velocity distribution. At large values of x, the integral in (2·1) may be expanded as an infinite series of solid harmonics, the coefficients of which are integral moments of the distribution of $\partial^2 u_i(\mathbf{x}')u_j(\mathbf{x}')/\partial x_i' \partial x_j'$. All these integrals converge at the instant under consideration, so that the expansion is justified. Thus, in tensor notation,

$$p(\mathbf{x}) \sim \frac{1}{x}\int A(\mathbf{x}')\,d\mathbf{x}' - \frac{\partial}{\partial x_i}\left(\frac{1}{x}\right)\int x_i' A(\mathbf{x}')\,d\mathbf{x}' + \frac{1}{2!}\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{x}\right)\int x_i' x_j' A(\mathbf{x}')\,d\mathbf{x}' + \dots, \tag{2·2}$$

for large values of x, where $A(\mathbf{x}')$ stands for $(4\pi)^{-1}\,\partial^2 u_i(\mathbf{x}')\,u_j(\mathbf{x}')/\partial x_i' \partial x_j'$. In general, therefore, the pressure falls off as an integral power of x^{-1} at large values of x, and the pressure gradients produce a distribution of acceleration with the same property. Hence a flow in which the velocity is exponentially small at large values of x is in general possible only instantaneously, the subsequent asymptotic form of the velocity being an integral power-law in x^{-1}.

It may be the case, of course, that all coefficients in the expansion (2·2) vanish at the initial instant. Under such circumstances, the initial acceleration would also be exponentially small at large values of x. But, by virtue of the integral nature of the expression (2·1) for the pressure, any assumption which involves the idea that this situation persists for all time clearly imposes a severe dynamical restriction on the motion as a whole. It is essentially for this reason that fluid flows in which the velocity is exponentially small at large values of x are dynamically very special, and appear to be almost entirely restricted to problems of parallel flow and the closely analogous rotary motions.

The relevance of these remarks to our problem in homogeneous turbulence is clear. They strongly suggest that if homogeneous turbulence is initially such that statistical connexions over large distances are exponentially small, an effect of the fluctuating pressure field is to establish power-law forms of velocity correlation tensors at large values of r. Support for this suggestion is provided by the fact that the origin of the term in r^{-4} in Proudman & Reid's analysis of $S_{ijk}(\mathbf{r})$ may be traced solely to the pressure term in the Navier–Stokes equation. The suggestion also readily explains why Proudman & Reid did not find a power-law form of the velocity covariance $R_{ij}(\mathbf{r})$ in isotropic turbulence. The rate of change of $R_{ij}(\mathbf{r})$ involves the pressure explicitly only in the form of the pressure-velocity covariance $P_i(\mathbf{r})$, and in isotropic turbulence this tensor vanishes identically as a result of the symmetry and continuity conditions.

It is also of interest to examine the effect of the dynamical processes on the large-scale structure of the vorticity distribution. Returning to the simpler type of flow discussed above, we consider the distribution of the rate of change of the vorticity $\boldsymbol{\omega}(\mathbf{x})$ at an instant at which $|\boldsymbol{\omega}(\mathbf{x})|$ is assumed to be exponentially small at large values of x, although $|\mathbf{u}(\mathbf{x})|$ may be (and, according to the above discussion, will be, in general) only algebraically small. Now, in the dynamical equation for the vorticity, namely

$$\frac{\partial \omega_i}{\partial t} = \omega_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial \omega_i}{\partial x_j} + \nu \nabla^2 \omega_i, \qquad (2\cdot 3)$$

the presence of the velocity vector ensures that the rate of change of vorticity at any point depends on the entire vorticity distribution, by virtue of the kinematical result

$$\mathbf{u}(\mathbf{x}) = \operatorname{curl} \frac{1}{4\pi} \int \boldsymbol{\omega}(\mathbf{x}') \frac{d\mathbf{x}'}{|\mathbf{x}'-\mathbf{x}|}.$$

In this respect, the dynamical equations for the vorticity and velocity are similar. The important difference lies in the fact that every term on the right side of (2·3) involves the local vorticity in a multiplicative manner, so that the local conditions place a limit on the order of magnitude of the rate of change of vorticity. Thus, at the instant under consideration, the rate of change of vorticity is exponentially small at large values of x, and the same is obviously true of time derivatives of $\boldsymbol{\omega}(\mathbf{x})$ of all orders. Hence the assumption that the vorticity is exponentially small at large values of x imposes a very weak dynamical restriction on the motion, which amounts to little more than a specification of suitable initial conditions. Indeed, vorticity distributions with this particular property are quite common, and include most of the text-book examples of the motion of real fluids in an infinite region without boundaries. The motion at large values of x in such cases is, of course, irrotational, and it is tempting to surmise that the velocity correlation tensors in homogeneous turbulence will have a corresponding irrotational structure at large values of r in cases in which the vorticity correlations are initially exponentially small. In this respect, however, the analogy appears to be very imperfect, as we shall see in §4.

Before leaving this simplified problem, it is worth noting that, in the absence of external forces, the total linear momentum of the flow must be constant, and that the existence of this dynamical invariant restricts the changes in the asymptotic form of the velocity distribution that can be brought about by pressure forces. The leading term in the expansion of an irrotational motion at large values of x is the dipole field

$$u_i(\mathbf{x}) = M_j \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{x}\right), \qquad (2\cdot 4)$$

where the coefficient M_j is related to the total linear momentum of the flow as follows:

$$M_j = \frac{3}{8\pi} \int u_j(\mathbf{x})\, d\mathbf{x}.$$

In the case in which $|\mathbf{u}(\mathbf{x})|$ is exponentially small at large values of x at the initial instant, M_j is initially zero, and remains zero, and the asymptotic form of $u_i(\mathbf{x})$ that is built up by pressure forces is such that $|\mathbf{u}(\mathbf{x})| = o(x^{-3})$ permanently.

The analogous concept in homogeneous turbulence seems to be the the 'linear momentum of an eddy', of which a suitable measure is the mean square momentum per unit volume of fluid, defined by

$$M_{ij} = \lim_{V\to\infty} \frac{1}{V} \overline{\int u_i(\mathbf{x})\, d\mathbf{x} \int u_j(\mathbf{x}')\, d\mathbf{x}'}$$

$$= \int R_{ij}(\mathbf{r})\, d\mathbf{r}. \qquad (2\cdot 5)$$

The volume integral in (2·5) is determined, in a manner analogous to (2·4), by the behaviour of $R_{ij}(\mathbf{r})$ at large values of r; for the continuity condition enables the integral to be expressed in the form

$$\int R_{ij}(\mathbf{r})\, \mathrm{d}r = \lim_{R\to\infty} \int_{r=R} r_i R_{kj}(\mathbf{r})\, \mathrm{d}S_k,$$

where the integration is over a sphere of large radius R, so that the integral is determined by

$$\lim_{r\to\infty} r^3 R_{ij}(\mathbf{r}). \tag{2·6}$$

From the analogy with linear momentum of a velocity field which vanishes at infinity, we might expect M_{ij} to be a dynamical invariant of the turbulence, and we shall see later that this is indeed the case, at any rate in the circumstances described in §3 in which M_{ij} is equal to zero. Thus when the initial conditions are such that M_{ij} vanishes we should expect the large-scale structure to have the property

$$R_{ij}(\mathbf{r}) = o(r^{-3})$$

for large values of r, permanently, and later work will confirm this.

The suggestive analogies presented in this section have made clear the nature of the problem before us. We have to examine both the conditions under which homogeneous turbulence is created and the subsequent effect of pressure forces. The former of these problems is considered in the following section, and then the effect of pressure forces is taken up in §4. Sections 5 and 6 are concerned with the properties of the resulting large-scale structure of the turbulence.

3. The hypothesis of convergent integral moments at an initial instant

We have explained, in §2, how many of the previously published results about the large-scale structure of homogeneous turbulence may be in error through being based on the assumption that integral moments of the velocity correlations converge. In its widest form this assumption, now seen not to be valid generally, would state that all integral moments of all cumulants* of the velocity distribution converge. The intuitive idea behind the assumption was that practical methods of generating homogeneous turbulence, such as the usual method of passing a stream of fluid through a regular array of rods, seem unlikely to be able to produce a turbulent motion with statistical connexions persisting over distances large compared with the length characteristic of the array of rods. It must be conceded that, despite its new-found lack of validity, some aspects of this idea have a certain plausibility, and the new hypothesis to be put forward here is in fact a modification of the old one.

Whatever the nature of the method of generating the homogeneous turbulence, we can regard the turbulence as having had its origin in the instability of some laminar flow system and in the growth of small velocity fluctuations superimposed on the laminar flow. Now, it is

* A single example will make clear the meaning of a cumulant of the velocity distribution; the general fourth-order cumulant is defined as

$$\overline{u_i u_j' u_k'' u_l'''} - \overline{u_i u_j'}.\overline{u_k'' u_l'''} - \overline{u_i u_k''}.\overline{u_j' u_l'''} - \overline{u_i u_l'''}.\overline{u_j' u_k''}.$$

We choose to talk about cumulants rather than mean values of velocity products, because the former approach zero when the distance between any two of the points at which the velocities are taken becomes large, whereas the latter may not do so.

widely believed (although anything in the nature of a rigorous proof is not available) that the statistical properties of a field of turbulence depend only on the boundary conditions appropriate to the laminar flow from which it develops, and not at all on the properties of the superimposed velocity fluctuations (provided that they are *small*); the role of the superimposed fluctuations seems to be simply to act as a trigger for the instability, and the turbulence develops its own statistical properties through the inertial interaction of the various Fourier components of the motion. (The implication for the case of homogeneous turbulence generated by a grid is that the statistical properties of the motion depend only on the shape of the grid of rods and not on the properties of *small* velocity fluctuations superimposed on the uniform incident stream.*) We know of no experimental evidence that conflicts with this idea and there is of course a great deal of evidence in its favour (for instance, the properties of fully-developed turbulent flow in a pipe seem to be independent of conditions in the inlet length). According to this notion of statistical uniqueness of the turbulence, the statistical properties of the fluctuation superimposed on the laminar flow can be chosen arbitrarily, and in particular can be chosen to be such that all integral moments of cumulants of the small fluctuating velocity converge. (It is clear from the remarks of §2 that when the intensity of homogeneous turbulence is small, and inertia and pressure forces are negligible, convergence of integral moments at any one instant implies convergence at subsequent instants, so that this property of the superimposed disturbance is not transitory.)

With this idea in mind, we believe it is permissible to assume that there is an initial instant, in the history of turbulence generated by a grid, at which integral moments of cumulants of the velocity distribution converge. At this initial instant the motion is not a fully developed turbulence in the usual sense, and its only claim to be regarded as requiring statistical specification lies in the random character of the small disturbance superimposed on the laminar flow. At times subsequent to this initial instant the small disturbance is amplified until it becomes 'finite', fully-developed turbulence forms through inertial interaction of Fourier components, and finally, by a process of diffusion, the turbulence become homogeneous (in the cases under consideration here). We shall see in the following section that power-law forms in the velocity correlations in homogeneous turbulence develop whenever the Reynolds number is not small. Thus it seems likely that with practical methods of generating homogeneous turbulence, power-law forms of the velocity correlations will develop before the turbulence has become properly homogeneous. Nevertheless, the hypothesis we propose to make here idealizes this complicated process of initial development, and supposes that the large-scale structure of homogeneous turbulence developed by a grid (or any other regular array of obstacles or holes) in a stream is the same *as if* it had always been homogeneous and had developed from some initial state in which all integral moments of cumulants of the velocity distribution converged. We are substituting a relatively simple model for the rather complex real process by which homogeneous turbulence is generated, in the belief that the model retains all the essential features of the real situation.

* Homogeneous turbulence generated by placing a grid in a stream carrying velocity fluctuations that are not small are excluded from our investigation, since the statistical properties of the velocity fluctuations approaching the grid would then need to be specified as part of the boundary conditions for the turbulence generated downstream of the grid and the problem would be too general to be tractable.

There are two separate ideas contained in our hypothesis. One is that the properties of the small disturbance that triggers off the instability of the laminar flow near the grid have no influence on the turbulence. It is not pretended that the cumulants of the distribution of velocity in the small disturbance approaching the grid are exponentially small in reality; rather, the assumption is that the long-range statistical connexions possessed by the small disturbance (whether of a power-law, or any other, form) are negligible compared with those that are subsequently developed by pressure forces when the turbulent velocity fluctuations are large. The other idea is that although the process of turbulent diffusion, by which the turbulence becomes homogeneous at some small distance downstream from the grid, and the process of development of long-range statistical connexions by pressure forces, proceed simultaneously, the end result is the same (so far as the functional forms of the statistical quantities are concerned) as if the latter process did not begin until the former was complete.

It may be remarked that in the absence of this hypothesis, or something closely related to it, it seems to be impossible to make any theoretical deductions at all about the large-scale structure of the turbulence, and correspondingly difficult to account for the known experimental results about the final period of decay. Quite apart from considerations of the initial conditions from which the turbulence may have developed, we have been unable to find any large-scale structure that remains unchanged in form by dynamical action, other than the large-scale structure that develops from the initial condition described above. The consequence of the hypothesis that is at once so useful mathematically—in that absolute intractability is avoided—and, seemingly, so necessary if the observed rate of decay in the final period is to be accounted for, is that the velocity correlations develop *integral* power-law forms.

In view of its importance for the remainder of the paper, let us repeat our hypothesis:

Homogeneous turbulence that is generated by placing a grid in a uniform stream carrying small velocity fluctuations has a large-scale structure like that which would develop, by dynamical action, in a field of turbulence which at some initial instant t_0 *is homogeneous and has convergent integral moments of cumulants of the velocity distribution.*

4. The generation of power-law forms of velocity cumulants at large values of the separation

In this section we shall make use of the hypothesis of the preceding section to determine the forms of mean values of products of velocities at points separated by large distances, and at times subsequent to the initial instant t_0. Our procedure is to determine the asymptotic form, at large values of the separation r, of the various time derivatives of mean values at the initial instant t_0. Then if the mean value at time t is written as a Taylor series in $t-t_0$, its asymptotic form at large r will be the same as that of the time derivative with the largest asymptotic form at $t=t_0$. This procedure gives a definite result only for values of $t-t_0$ such that the infinite Taylor series in $t-t_0$ is valid (the existence of all time derivatives of mean values of velocity products may reasonably be taken for granted, but the vanishing of Lagrange's form of the remainder for arbitrarily large values of $t-t_0$ is an open question), but we think it is highly unlikely that such a fundamental property as the asymptotic form of a mean value could change at some finite value of $t-t_0$. So far as we have been able to tell from an examination of time derivatives of mean values at an instant subsequent to $t = t_0$, the asymptotic forms that are established in this section persist and are unchanged (in form) by further dynamical action.

We proceed to a consideration of time derivatives of the most important of the mean values of velocity products, namely, $R_{ij}(\mathbf{r})$, and begin with the well-known exact expression for the first time derivative:

$$\frac{\partial R_{ij}(\mathbf{r})}{\partial t} = \frac{\partial}{\partial r_k}(\overline{u_i u_k u_j'} - \overline{u_i u_k' u_j'}) + \frac{\partial \overline{p u_j'}}{\partial r_i} - \frac{\partial \overline{p' u_i}}{\partial r_j} + 2\nu \nabla^2 R_{ij}(\mathbf{r}), \tag{4·1}$$

where $u_j' = u_j(\mathbf{x}')$ and $\mathbf{r} = \mathbf{x}' - \mathbf{x}$. At the initial instant t_0, in accordance with the hypothesis of the preceding section, $R_{ij}(\mathbf{r})$ is so small at large values of r that all its integral moments with respect to $\mathbf{r}$ converge. This is also true of the inertia and viscous force terms on the right-hand side of (4·1). But taking the divergence of (4·1) shows that

$$\nabla^2 \overline{p u_j'} = -\frac{\partial^2 \overline{u_i u_k u_j'}}{\partial r_i \partial r_k},$$

so that

$$\overline{p u_j'} = \frac{1}{4\pi} \int \frac{\partial^2 \overline{u_i'' u_k'' u_j'}}{\partial x_i'' \partial x_k''} \frac{d\mathbf{x}''}{|\mathbf{x}'' - \mathbf{x}|}, \tag{4·2}$$

and this is not a quantity whose exponential smallness at large values of r at the instant t_0 is guaranteed by the hypothesis. The hypothesis does ensure, however, that all integral moments of $\overline{u_i'' u_k'' u_j'}$ exist at $t = t_0$, so that it is possible to expand the integral in inverse powers of r, the expansion having a form like that in (2·2). The coefficients of terms of order r^{-1} and r^{-2} are zero because of the double derivative in the integrand of (4·2),* and since $\overline{u_i'' u_k'' u_j'}$ is solenoidal (with respect to $\mathbf{x}'' - \mathbf{x}'$) in the j-index the coefficient of the term of order r^{-3} is also zero. Hence, as $r \to \infty$,

$$\overline{p u_j'} \sim -\frac{1}{4\pi} \frac{\partial^3}{\partial r_i \partial r_k \partial r_l}\left(\frac{1}{r}\right) \int \overline{u_i'' u_k'' u_j'}\, s_l \, d\mathbf{s} \tag{4·3}$$

(where $\mathbf{s} = \mathbf{x}'' - \mathbf{x}'$), there being no kinematical reason why the integral coefficient in this term should vanish; consequently, as $r \to \infty$,

$$\left[\frac{\partial R_{ij}(\mathbf{r})}{\partial t}\right]_{t_0} \sim -T_{kljm} \frac{\partial^4}{\partial r_i \partial r_k \partial r_l \partial r_m}\left(\frac{1}{r}\right) - T_{klim} \frac{\partial^4}{\partial r_j \partial r_k \partial r_l \partial r_m}\left(\frac{1}{r}\right), \tag{4·4}$$

where

$$T_{kljm} = \frac{1}{4\pi} \int \overline{u_k'' u_l'' u_j'}\, s_m \, d\mathbf{s}. \tag{4·5}$$

Thus the effect of pressure forces, which do not have a local origin and which depend on the whole distribution of velocity, is to build up the velocity covariance so that it decreases at infinity not more rapidly than r^{-5}. It will be noticed that the two pressure terms in (4·1) are irrotational in the i and j indices, respectively, so that $(\partial \overline{\omega_i \omega_j'}/\partial t)_{t_0}$ is exponentially small as $r \to \infty$, corresponding to the fact that pressure forces have no direct effect on vorticity. This smallness of the vorticity persists in all the time derivatives in the case of a bounded field of vorticity, but in the case of homogeneous turbulence it will appear that the *indirect*

* In both cases the volume integrals are converted to surface integrals and use is made of the fact that $\overline{u_i u_k u_j'}$ is of smaller order than r^{-3}.

effect of pressure forces, acting through the velocity field, can build up power-law forms of the vorticity covariance at times subsequent to t_0. The point is that the irrotational velocity field associated with, and surrounding, a correlated region of vorticity has an effect (through stretching of vortex lines, for example) on the vorticity at points distant from the correlated region, and in this way a statistical connexion between the vorticities at distant points can be developed. To see this, we must examine further time derivatives of $R_{ij}(\mathbf{r})$.

It is scarcely feasible to write down the complete expressions for $\partial^2 R_{ij}(\mathbf{r})/\partial t^2$, $\partial^3 R_{ij}(\mathbf{r})/\partial t^3$, etc., so that we propose to describe the procedure and to exhibit the results without going through the algebra in detail. First the expressions for $\partial^2 R_{ij}(\mathbf{r})/\partial t^2$ and $\partial^3 R_{ij}(\mathbf{r})/\partial t^3$ are obtained from (4·1) and are cleared of time derivatives by substitution from the Navier–Stokes equation. The resulting terms that contain the pressure (these being the only ones that will not be exponentially small at $t = t_0$) are then examined with the object of deriving asymptotic forms like (4·3). In this way the following orders of magnitude at large r of the first three time derivatives of $\overline{u_i u_j'}$ at $t = t_0$ are found. Orders of magnitude of the corresponding time derivatives of $\overline{\omega_i u_j'}$ and $\overline{\omega_i \omega_j'}$ have been found by discarding the relevant irrotational terms in the expressions for $\partial R_{ij}(\mathbf{r})/\partial t$, $\partial^2 R_{ij}(\mathbf{r})/\partial t^2$ and $\partial^3 R_{ij}(\mathbf{r})/\partial t^3$:

	$\overline{u_i u_j'}$	$\overline{\omega_i u_j'}$	$\overline{\omega_i \omega_j'}$
function	exp. small	exp. small	exp. small
first time derivative	r^{-5}	r^{-6}	exp. small
second time derivative	r^{-5}	r^{-6}	exp. small
third time derivative	r^{-5}	r^{-6}	r^{-8}

A power-law dependence on r appears in the first time derivative of $\overline{\omega_i u_j'}$; this is the correlation between the vorticity at $\mathbf{x}$, and the irrotational acceleration at $\mathbf{x}'$ associated with a correlated region of vorticity centred on $\mathbf{x}$, as already explained. A power-law dependence on r appears in $\overline{\omega_i \omega_j'}$ when we come to the third time derivative, and it does so through the indirect effect of pressure forces on the vorticity, although a detailed statement of the process cannot be given. There is little doubt that power-law asymptotic forms of *all* cumulants of the velocity and its spatial derivatives do develop—if sufficient time derivatives are taken—as a result of the direct or indirect effect of pressure forces. The important results in the above table are that $\overline{u_i u_j'}$ is at least of order r^{-5} for large values of r, at times subsequent to t_0, and that $\overline{\omega_i \omega_j'}$ is at least of order r^{-8}. We proceed now to show, from a consideration of the nth time derivative of $\overline{u_i u_j'}$ and $\overline{\omega_i \omega_j'}$ that in fact no terms of larger order ever appear.

The nth time derivative of $\overline{u_i u_j'}$ can be written as

$$\frac{\partial^n \overline{u_i u_j'}}{\partial t^n} = \sum_{m=0}^{n} C(m) \overline{\frac{\partial^m u_i}{\partial t^m} \frac{\partial^{n-m} u_j'}{\partial t^{n-m}}}, \tag{4·6}$$

where $C(m)$ is a number. Now the largest possible asymptotic form of any mean value in the present context is possessed by mean values of the kind

$$\overline{p(\mathbf{x})\, F(\mathbf{x}')} \sim \frac{1}{4\pi} \frac{\partial^2}{\partial r_k \partial r_l} \left(\frac{1}{r}\right) \int \overline{u_k'' u_l'' F(\mathbf{x}')}\, \mathrm{d}\mathbf{x}'';$$

if $F(\mathbf{x}')$ is a solenoidal tensor this integral vanishes and $\overline{p(\mathbf{x})\, F(\mathbf{x}')}$ is of order r^{-4} when r is large. It is not difficult to show that $\overline{\frac{\partial p(\mathbf{x})}{\partial t} F(\mathbf{x}')}$ is of smaller order than $\overline{p(\mathbf{x})\, F(\mathbf{x}')}$ by one power of r (essentially because elimination of a time derivative by means of the Navier–Stokes equation introduces one space derivative). Hence the largest asymptotic form that can arise from the summation (4·6)* occurs in the term $m = 1$ (or $m = n-1$) and is

$$C(1)\frac{\partial}{\partial r_i}\left(\overline{p\frac{\partial^{n-1}u_j'}{\partial t^{n-1}}}\right) = O(r^{-5}),$$

since $\partial^{n-1}u_j'/\partial t^{n-1}$ is a solenoidal vector. This establishes that when r is large $R_{ij}(\mathbf{r})$ is of order r^{-5} at all times subsequent to the instant t_0.

Turning now to the vorticity covariance $\overline{\omega_i\omega_j'}$, if we omit terms that vanish when the curl with respect to both the i and j indices is taken and terms containing viscosity (since power-law forms arising from the terms containing ν are clearly of smaller order than those arising from terms not containing ν), the right-hand side of (4·6) becomes

$$-\overline{u_i\frac{\partial^{n-1}}{\partial t^{n-1}}\left(\frac{\partial u_j'u_k'}{\partial x_k'}\right)}-\overline{u_j'\frac{\partial^{n-1}}{\partial t^{n-1}}\left(\frac{\partial u_iu_k}{\partial x_k}\right)}+\sum_{m=1}^{n-1}C(m)\overline{\frac{\partial^{m-1}}{\partial t^{m-1}}\left(\frac{\partial u_iu_k}{\partial x_k}\right)\frac{\partial^{n-m-1}}{\partial t^{n-m-1}}\left(\frac{\partial u_j'u_l'}{\partial x_l'}\right)}$$
$$=-\frac{\partial}{\partial r_k}\left(\overline{u_i\frac{\partial^{n-1}u_j'u_k'}{\partial t^{n-1}}}-\overline{u_j'\frac{\partial^{n-1}u_iu_k}{\partial t^{n-1}}}\right)-\frac{\partial^2}{\partial r_k\partial r_l}\sum_{m=1}^{n-1}C(m)\overline{\frac{\partial^{m-1}u_iu_k}{\partial t^{m-1}}\frac{\partial^{n-m-1}u_j'u_l'}{\partial t^{n-m-1}}}. \qquad (4·7)$$

Considerations similar to those just used show that the largest asymptotic form that can arise from any term in the summation on the right-hand side of (4·7) is of order r^{-4}, and the corresponding contributions to $\partial^n\overline{u_iu_j'}/\partial t^n$ and $\partial^n\overline{\omega_i\omega_j'}/\partial t^n$ are of order r^{-6} and r^{-8} respectively. Terms of the kind $\overline{p(\mathbf{x})F(\mathbf{x}')}$ that arise from the expression in circular brackets on the right-hand side of (4·7) are necessarily such that $F(\mathbf{x}')$ is the solenoidal vector $u_j(\mathbf{x}')$ and are therefore of order r^{-4}; pressure enters always as a pressure gradient so that again there are contributions to $\partial^n\overline{u_iu_j'}/\partial t^n$ and $\partial^n\overline{\omega_i\omega_j'}/\partial t^n$ of order r^{-6} and r^{-8} respectively, and the result about $\overline{\omega_i\omega_j'}$ is established.

The conclusion, on which all the analysis in later sections is based, is that, if the hypothesis of §3 be allowed, the velocity covariance is in general of order r^{-5} at large values of r and the vorticity covariance is in general of order r^{-8}, at all times subsequent to the initial instant $t=t_0$.

Similar results can be found for other mean values, and in particular for the important triple product $S_{ikj}(\mathbf{r}) = \overline{u_iu_ku_j'}$ which occurs in the dynamical equation for $\overline{u_iu_j'}$. In the expression for the first time derivative of $\overline{u_iu_ku_j'}$ there appears the term

$$-\overline{u_iu_k\frac{\partial p'}{\partial x_j'}} = -\frac{1}{4\pi}\frac{\partial}{\partial r_j}\int\overline{u_iu_k\frac{\partial^2u_l''u_m''}{\partial x_l''\partial x_m''}}\frac{d\mathbf{x}''}{|\mathbf{x}''-\mathbf{x}'|} \sim -\frac{1}{4\pi}\frac{\partial^3}{\partial r_j\partial r_l\partial r_m}\left(\frac{1}{r}\right)\int\overline{u_iu_k(u_l''u_m''-\overline{u_l''u_m''})}\,d\mathbf{x}'',$$

as $r\to\infty$. No larger asymptotic form appears in any higher-order time derivatives, so that $\overline{u_iu_ku_j'}$ is in general of order r^{-4} at large values of r at times subsequent to $t=t_0$. Moreover, consideration of the nth time derivative of $\overline{u_iu_ku_j'}$ at $t=t_0$, in the manner described above, shows that the term of order r^{-4} in a time derivative of $\overline{u_iu_ku_j'}$ is always such that its curl in the j-index is zero, and that the largest term whose curl does not vanish is of order r^{-5}.

* Use is made of the fact that the asymptotic form of the mean value $\overline{p(\mathbf{x})F(\mathbf{x})G(\mathbf{x}')}$, as $r\to\infty$, is

$$\frac{1}{4\pi}\frac{\partial^2}{\partial r_k\partial r_l}\left(\frac{1}{r}\right)\left\{\overline{F(\mathbf{x})}\int\overline{(u_k''u_l''-\overline{u_k''u_l''})G(\mathbf{x}')}\,d\mathbf{x}''+\overline{G(\mathbf{x}')}\int\overline{(u_k''u_l''-\overline{u_k''u_l''})F(\mathbf{x})}\,d\mathbf{x}''\right\},$$

where, again, if $F(\mathbf{x})$ and $G(\mathbf{x}')$ are solenoidal, the integrals vanish and the next term in the expansion in powers of r^{-1} must be taken.

5. Kinematical structure of the large-scale motion

The task is now to find the detailed kinematical structure of the large-scale motion, since a knowledge of this structure is essential for the dynamical investigations which follow. Mathematically, the problem is that of determining the asymptotic forms of both the spectrum tensor $\Phi_{ij}(\mathbf{k})$ as $k \to 0$, and the velocity covariance $R_{ij}(\mathbf{r})$ as $r \to \infty$. Available for this calculation are the results, obtained in the preceding section, that $R_{ij}(\mathbf{r}) = O(r^{-5})$ and

$$V_{ij}(\mathbf{r}) = \overline{\omega_i(\mathbf{x})\,\omega_j(\mathbf{x}+\mathbf{r})} = O(r^{-8}) \tag{5·1}$$

for large values of r, together with the incompressibility condition.

To facilitate comparison with the earlier investigations, we begin with the immediate consequences of the convergence, or otherwise, of the integral moments of $R_{ij}(\mathbf{r})$, when the order of magnitude of this tensor at large values of r is that stated above. Thus, the absolute convergence of the integrals

$$\int R_{ij}(\mathbf{r})\,d\mathbf{r} \quad \text{and} \quad \int r_k R_{ij}(\mathbf{r})\,d\mathbf{r}$$

ensures (using the known properties of Fourier transforms) that the tensors $\Phi_{ij}(\mathbf{k})$ and $\partial\Phi_{ij}(\mathbf{k})/\partial k_k$ are continuous functions of $\mathbf{k}$ for all values of $\mathbf{k}$, including $\mathbf{k} = 0$. The second derivative $\partial^2\Phi_{ij}(\mathbf{k})/\partial k_k \partial k_l$, on the other hand, which is given formally by the expression

$$-\frac{1}{8\pi^3}\int r_k r_l R_{ij}(\mathbf{r})\,e^{-i\mathbf{k}\cdot\mathbf{r}}\,d\mathbf{r} \tag{5·2}$$

exists as a continuous function of $\mathbf{k}$ at all non-zero values of $\mathbf{k}$, by virtue of the convergence of the integral

$$\int |\, r_k r_l R_{ij}(\mathbf{r})\,|^2\,d\mathbf{r},$$

but is singular at $\mathbf{k} = 0$ since the integral

$$\int |\, r_k r_l R_{ij}(\mathbf{r})\,|\,d\mathbf{r} \tag{5·3}$$

diverges. The divergence of (5·3) is logarithmic, and since the exponential factor in (5·2) is effectively unity for values of $\mathbf{r}$ within a sphere of radius $O(k^{-1})$, it follows that the singularity in (5·2) at $\mathbf{k} = 0$ is not worse than logarithmic, i.e. in general,

$$\frac{\partial^2\Phi_{ij}(\mathbf{k})}{\partial k_k \partial k_l} = O(\ln k) \tag{5·4}$$

for small values of k.

If, now, the spectrum tensor $\Phi_{ij}(\mathbf{k})$ be expanded in a Taylor series with remainder,

$$\Phi_{ij}(\mathbf{k}) = \Phi_{ij}(0) + k_k\left(\frac{\partial\Phi_{ij}(\mathbf{k})}{\partial k_k}\right)_0 + \frac{1}{2!}k_k k_l\left(\frac{\partial^2\Phi_{ij}(\mathbf{k})}{\partial k_k \partial k_l}\right)_{\alpha\mathbf{k}} \tag{5·5}$$

where $0 < |\,\alpha\,| \leqslant 1$, we obtain the equation

$$8\pi^3\Phi_{ij}(\mathbf{k}) = \int R_{ij}(\mathbf{r})\,d\mathbf{r} - ik_k\int r_k R_{ij}(\mathbf{r})\,d\mathbf{r} + O(k^2 \ln k) \tag{5·6}$$

for small values of k. The spectral form of the incompressibility condition

$$k_i\Phi_{ij}(\mathbf{k}) = 0,$$

then yields the result

$$\int R_{ij}(\mathbf{r})\,\mathrm{d}\mathbf{r} = 0, \tag{5·7}$$

and Cramer's theorem that $X_i \overset{*}{X}_j \Phi_{ij}(\mathbf{k})$ must be non-negative for an arbitrary choice of the complex vector X_i (the asterisk here denotes the complex conjugate), yields the further result

$$\int r_k R_{ij}(\mathbf{r})\,\mathrm{d}\mathbf{r} = 0. \tag{5·8}$$

So far as the first two terms of the expansion (5·5) are concerned, therefore, the analysis is identical with that given by Batchelor (1953).

The important difference between the present results and earlier work is the fact that the leading term in the expansion of the spectrum tensor $\Phi_{ij}(\mathbf{k})$ at small values of k now involves a singularity whose nature is determined by the local behaviour of the velocity covariance $R_{ij}(\mathbf{r})$ at large values of r. If nothing is known about the latter behaviour, other than its general order of magnitude, it does not seem to be possible to derive any further information about the singularity in the spectrum, apart from one or two results concerned with its tensorial character that follow from the incompressibility condition. In particular, it does not seem to be possible to improve upon the estimate (5·4) for the order of magnitude of the second derivative of $\Phi_{ij}(\mathbf{k})$ at small values of k. However, there is also available the associated result that the vorticity covariance $V_{ij}(\mathbf{r})$ is $O(r^{-8})$ for large values of r, and it will appear that this second result determines the entire asymptotic structure of $R_{ij}(\mathbf{r})$ at large values of r, and therefore the precise nature of the singularity in the spectrum tensor.

Considering the asymptotic behaviour of $R_{ij}(\mathbf{r})$ first, the relation between this tensor and $V_{ij}(\mathbf{r})$ in an incompressible fluid is (see Batchelor 1953)

$$\nabla^2 R_{ij}(\mathbf{r}) = V_{ji}(\mathbf{r}) - \delta_{ij} V_{kk}(\mathbf{r}) - \frac{\partial^2 R_{kk}(\mathbf{r})}{\partial r_i \partial r_j}, \tag{5·9}$$

of which a special case is

$$\nabla^2 R_{kk}(\mathbf{r}) = -V_{kk}(\mathbf{r}). \tag{5·10}$$

The general solution of (5·10) is

$$R_{kk}(\mathbf{r}) = \frac{1}{4\pi}\int V_{kk}(\mathbf{r}')\frac{\mathrm{d}\mathbf{r}'}{|\mathbf{r}'-\mathbf{r}|}, \tag{5·11}$$

and, for large values of r, the integral may be expanded in powers of r^{-1} with coefficients which are integral moments of $V_{kk}(\mathbf{r})$, in the manner described in §2. Since $V_{kk}(\mathbf{r}) = O(r^{-8})$ for large values of r, the expansion is valid up to, and including, the term in r^{-5}, the remainder then being of order r^{-6}. Moreover, $R_{kk}(\mathbf{r}) = O(r^{-5})$ for large values of r, so there is just one valid non-zero term in the expansion. Thus, writing

$$C_{ijlmnp} = \frac{1}{4\pi}\frac{1}{4!}\int r_l r_m r_n r_p V_{ij}(\mathbf{r})\,\mathrm{d}\mathbf{r}, \tag{5·12}$$

we have

$$R_{kk}(\mathbf{r}) = C_{kklmnp}\frac{\partial^4}{\partial r_l \partial r_m \partial r_n \partial r_p}\left(\frac{1}{r}\right) + O(r^{-6}). \tag{5·13}$$

In the general case of the uncontracted velocity covariance, the solution of equation (5·9) may also be written in the form of an integral analogous to (5·11). But the last term on the right-hand side of (5·9) is $O(r^{-7})$ for large values of r, so that the expansion of the integral cannot be carried as far as the term in r^{-5}. In other words, the only valid terms in the expansion are identically zero. This difficulty may be overcome in the following way.

The solution of (5·9) may be written in the form

$$R_{ij}(\mathbf{r}) = -\frac{1}{4\pi}\int[V_{ji}(\mathbf{r}') - \delta_{ij}V_{kk}(\mathbf{r}')]\frac{d\mathbf{r}'}{|\mathbf{r}'-\mathbf{r}|} - \frac{\partial^2\phi(\mathbf{r})}{\partial r_i \partial r_j}, \tag{5·14}$$

where

$$\nabla^2\phi(\mathbf{r}) = R_{kk}(\mathbf{r}) \tag{5·15}$$

and $\phi(\mathbf{r}) = O(r^{-3})$ at large values of r. Now let $\psi(\mathbf{r})$ be any function (with no singularities) which has the behaviour

$$\psi(\mathbf{r}) = \tfrac{1}{2}C_{kklmnp}\frac{\partial^4 r}{\partial r_l \partial r_m \partial r_n \partial r_p} + O(r^{-4})$$

for large values of r. The corresponding behaviour of $\nabla^2\psi(\mathbf{r})$ is

$$\nabla^2\psi(\mathbf{r}) = C_{kklmnp}\frac{\partial^4}{\partial r_l \partial r_m \partial r_n \partial r_p}\left(\frac{1}{r}\right) + O(r^{-6}),$$

and hence, from equations (5·13) and (5·15),

$$\nabla^2[\phi(\mathbf{r}) - \psi(\mathbf{r})] = O(r^{-6}) \tag{5·16}$$

for large values of r. Also, the general solution of the equation obtained by subtracting $\nabla^2\psi(\mathbf{r})$ from both sides of (5·15) is

$$\phi(\mathbf{r}) = \psi(\mathbf{r}) - \frac{1}{4\pi}\int[R_{kk}(\mathbf{r}') - \nabla^2\psi(\mathbf{r}')]\frac{d\mathbf{r}'}{|\mathbf{r}'-\mathbf{r}|},$$

and since the expansion of this integral as far as the term in r^{-3} is now justified by (5·16), we have

$$\phi(\mathbf{r}) = \tfrac{1}{2}C_{kklmnp}\frac{\partial^4 r}{\partial r_l \partial r_m \partial r_n \partial r_p} - \frac{1}{8\pi}\frac{\partial^2}{\partial r_l \partial r_m}\left(\frac{1}{r}\right)\int r_l' r_m'[R_{kk}(\mathbf{r}') - \nabla^2\psi(\mathbf{r}')]\,d\mathbf{r}' + O(r^{-4}) \tag{5·17}$$

for large values of r. Hence the asymptotic form of $R_{ij}(\mathbf{r})$ is, from (5·14) and (5·17),

$$\begin{aligned} R_{ij}(\mathbf{r}) = &-\tfrac{1}{2}C_{kklmnp}\frac{\partial^6 r}{\partial r_i \partial r_j \partial r_l \partial r_m \partial r_n \partial r_p} - (C_{ijlmnp} - \delta_{ij}C_{kklmnp})\frac{\partial^4}{\partial r_l \partial r_m \partial r_n \partial r_p}\left(\frac{1}{r}\right) \\ &+\frac{1}{8\pi}\frac{\partial^4}{\partial r_i \partial r_j \partial r_l \partial r_m}\left(\frac{1}{r}\right)\int r_l' r_m'[R_{kk}(\mathbf{r}') - \nabla^2\psi(\mathbf{r}')]\,d\mathbf{r}' + O(r^{-6}). \end{aligned} \tag{5·18}$$

It remains to evaluate the volume integral in the result (5·18). Actually, the two terms in the integrand each give convergent integrals separately, as the possible logarithmic singularities do not occur. For if r^* is a value of r sufficiently large to permit the replacement of the integrands by their asymptotic forms, then the contribution to each of these integrals from large values of r is of the form

$$\int_{r>r^*} r_l r_m F(\mathbf{r})\,d\mathbf{r}, \tag{5·19}$$

where $F(\mathbf{r})$ is a known function of order r^{-5}. It follows that if $F(\mathbf{r})$ can be written as the third derivative of a function of order r^{-2}, then a threefold integration by parts enables (5·19) to be expressed entirely in terms of convergent surface integrals whose sum vanishes, so that the complete volume integral converges. The integrands in (5·18) clearly satisfy the required conditions (see (5·13) and the definition of $\psi(\mathbf{r})$). We may also note here a point of some importance in the sequel, namely that the integral

$$L_{ijmn} = \frac{1}{4\pi}\int r_m r_n R_{ij}(\mathbf{r})\,\mathrm{d}\mathbf{r}$$

converges by the same argument, since the asymptotic form of $R_{ij}(\mathbf{r})$ may, by (5·18), be written as a third derivative of a tensor of order r^{-2}.

Taking first the integral in $\psi(\mathbf{r})$, in (5·18), two integrations by parts give

$$\frac{\partial^2}{\partial r_l \partial r_m}\left(\frac{1}{r}\right)\int r'_l r'_m \nabla^2\psi(\mathbf{r}')\,\mathrm{d}\mathbf{r}' = \frac{\partial^2}{\partial r_l \partial r_m}\left(\frac{1}{r}\right)\lim_{R\to\infty}\int_{r'=R}\left[r'_l r'_m \frac{\partial\psi(\mathbf{r}')}{\partial r'_p} - \delta_{lp} r'_m \psi(\mathbf{r}') - \delta_{mp} r'_l \psi(\mathbf{r}')\right]\mathrm{d}S_p, \tag{5·20}$$

where $\mathrm{d}S_p$ is the p-component of a surface element at position $\mathbf{r}'$, and the integration is over a large sphere. Residual volume integrals do not occur in (5·20) after contraction with the harmonic tensor outside the integral (as is obviously necessary since these volume integrals are arbitrary). On substitution of the asymptotic form of $\psi(\mathbf{r}')$ the right-hand side of (5·20) becomes

$$-\frac{5}{2}\frac{\partial^2}{\partial r_l \partial r_m}\left(\frac{1}{r}\right) C_{kkpqrs}\int r'_l r'_m \frac{\partial^4 r'}{\partial r'_p \partial r'_q \partial r'_r \partial r'_s} r'\,\mathrm{d}\Omega(\mathbf{r}'), \tag{5·21}$$

where $\mathrm{d}\Omega(\mathbf{r}')$ is the solid angle subtended at the origin by a surface element at the point $\mathbf{r}'$. Isotropic surface integrals of the type appearing in (5·21) occur frequently in the analysis, and, to avoid lengthy digressions in the text, we have evaluated them in appendix A, to which reference should also be made for the notation. Thus, in the present case, we have

$$\int r'_l r'_m \frac{\partial^4 r'}{\partial r'_p \partial r'_q \partial r'_r \partial r'_s} r'\,\mathrm{d}\Omega(\mathbf{r}') = \frac{8\pi}{5}\left[\tfrac{1}{7}\delta_{lmpqrs} - \tfrac{1}{3}\delta_{lm}\delta_{pqrs}\right],$$

so that
$$\frac{\partial^2}{\partial r_l \partial r_m}\left(\frac{1}{r}\right)\int r'_l r'_m \nabla^2\psi(\mathbf{r}')\,\mathrm{d}\mathbf{r}' = -\frac{4\pi}{7}\frac{\partial^2}{\partial r_l \partial r_m}\left(\frac{1}{r}\right) C_{kkpqrs}\delta_{lmpqrs}$$
$$= -\frac{4\pi\,.\,4!}{14}\frac{\partial^2}{\partial r_l \partial r_m}\left(\frac{1}{r}\right) C_{kkpplm} = -\frac{1}{14}\frac{\partial^2}{\partial r_l \partial r_m}\left(\frac{1}{r}\right)\int r'_l r'_m r'^2 V_{kk}(\mathbf{r}')\,\mathrm{d}\mathbf{r}'$$
$$= \frac{1}{14}\frac{\partial^2}{\partial r_l \partial r_m}\left(\frac{1}{r}\right)\int r'_l r'_m r'^2 \nabla^2 R_{kk}(\mathbf{r}')\,\mathrm{d}\mathbf{r}'.$$

After two integrations by parts, this last result becomes

$$-\frac{9}{14}\frac{\partial^2}{\partial r_l \partial r_m}\left(\frac{1}{r}\right)\lim_{R\to\infty}\int_{r'=R} r'_l r'_m r'_p R_{kk}(\mathbf{r}')\,\mathrm{d}S_p + \frac{\partial^2}{\partial r_l \partial r_m}\left(\frac{1}{r}\right)\int r'_l r'_m R_{kk}(\mathbf{r}')\,\mathrm{d}\mathbf{r}',$$

in which the surface integral vanishes in view of (5·13). Hence

$$\frac{\partial^2}{\partial r_l \partial r_m}\left(\frac{1}{r}\right)\int r'_l r'_m \left[R_{kk}(\mathbf{r}') - \nabla^2\psi(\mathbf{r}')\right]\mathrm{d}\mathbf{r}' = 0,$$

and the asymptotic behaviour of $R_{ij}(\mathbf{r})$ is, from (5·18),

$$R_{ij}(\mathbf{r}) = -\tfrac{1}{2}C_{kklmnp}\frac{\partial^6 r}{\partial r_i \partial r_j \partial r_l \partial r_m \partial r_n \partial r_p} - (C_{ijlmnp} - \delta_{ij}C_{kklmnp})\frac{\partial^4}{\partial r_l \partial r_m \partial r_n \partial r_p}\left(\frac{1}{r}\right) + O(r^{-6}). \tag{5·22}$$

We have not yet considered the consequences, so far as the tensor C_{ijlmnp} is concerned, of the fact that the vorticity is a solenoidal vector. This may be done by writing the condition in the form

$$\int r_l r_m r_n r_p r_j \frac{\partial V_{iq}(\mathbf{r})}{\partial r_q}\,d\mathbf{r} = 0,$$

and integrating once by parts. The surface integral vanishes, since $V_{iq}(\mathbf{r}) = O(r^{-8})$ for large values of r, and the volume integrals yield the condition

$$\sum_{\text{perm. } j,\, l,\, m,\, n,\, p} C_{ijlmnp} = 0. \tag{5·23}$$

In appendix B, the general solution of equation (5·23) is shown to be

$$C_{ijlmnp} = \frac{1}{4!} \sum_{\text{perm. } l,\, m,\, n,\, p}\ \sum_{\text{perm. } m,\, n,\, p} \epsilon_{ilq}\,\epsilon_{jmr}\,C_{qrnp}, \tag{5·24}$$

where C_{qrnp} is symmetric in the indices q, r, and n, p, and the contracted form C_{qrrp} vanishes. The inverse relation is

$$\begin{aligned} C_{lmnp} &= \tfrac{6}{5}\,\epsilon_{qrl}\,\epsilon_{stm}\,C_{qsrtnp} \\ &= \frac{1}{80\pi}\,\epsilon_{qrl}\,\epsilon_{stm}\int r_r r_t r_n r_p V_{qs}(\mathbf{r})\,d\mathbf{r}. \end{aligned} \tag{5·25}$$

It should also be noted that, by imposing the solenoidal condition on the vorticity covariance, we automatically impose the same condition on the velocity covariance, since the analysis depends on the validity of the basic equation (5·9) at all points of space. So it is that, when the formula (5·22) is written in terms of the new tensor C_{lmnp}, the result is the solenoidal behaviour

$$\begin{aligned} R_{ij}(\mathbf{r}) &= \tfrac{1}{4}C_{lmnp}\frac{\partial^6 r}{\partial r_i \partial r_j \partial r_l \partial r_m \partial r_n \partial r_p} - \tfrac{1}{2}C_{ilmn}\frac{\partial^4}{\partial r_j \partial r_l \partial r_m \partial r_n}\left(\frac{1}{r}\right) - \tfrac{1}{2}C_{jlmn}\frac{\partial^4}{\partial r_i \partial r_l \partial r_m \partial r_n}\left(\frac{1}{r}\right) + O(r^{-6}) \\ &= \tfrac{1}{4}C_{lmnp}\left(\delta_{il}\nabla^2 - \frac{\partial^2}{\partial r_i \partial r_l}\right)\left(\delta_{jm}\nabla^2 - \frac{\partial^2}{\partial r_j \partial r_m}\right)\frac{\partial^2 r}{\partial r_n \partial r_p} + O(r^{-6}), \end{aligned} \tag{5·26}$$

and all the kinematical conditions of the problem are satisfied.

It remains only for us to point out the relation between the tensor C_{lmnp} and the second integral moment of the velocity covariance L_{lmnp}. The calculation is rather long, and it must suffice here to indicate its nature and to quote the result. Starting from the relation (5·25), the tensor $V_{qs}(\mathbf{r})$ is written as a second derivative of $R_{ij}(\mathbf{r})$, and a double integration by parts then yields a collection of surface and volume integrals. All the surface integrals depend only on the asymptotic behaviour (5·26) of $R_{ij}(\mathbf{r})$, and may therefore be evaluated entirely in terms of C_{lmnp}. All the volume integrals are particular combinations of the tensor L_{lmnp}. Thus, formally, equation (5·25) is a relation between these two tensors, though it is not clear how much information is yielded by the relation until its detailed tensorial structure has been found. In fact, however, the relation determines each tensor uniquely in terms of the other, the equation for C_{lmnp} being

$$\begin{aligned} 20C_{lmnp} = &-\tfrac{92}{3}L_{lmnp} - \tfrac{8}{3}L_{nplm} - \tfrac{20}{3}(L_{lnmp} + L_{lpmn} + L_{mnlp} + L_{mpln}) \\ &+ \tfrac{23}{21}\delta_{lm}(L_{kknp} - L_{npkk}) + \tfrac{2}{21}\delta_{np}(L_{kklm} - L_{lmkk}) \\ &+ \tfrac{1}{21}\delta_{mp}(23L_{kkln} - 65L_{lnkk}) + \tfrac{1}{21}\delta_{mn}(23L_{kklp} - 65L_{lpkk}) \\ &+ \tfrac{1}{21}\delta_{lp}(23L_{kkmn} - 65L_{mnkk}) + \tfrac{1}{21}\delta_{ln}(23L_{kkmp} - 65L_{mpkk}) \\ &+ \tfrac{1}{2}L_{iikk}(\delta_{ln}\delta_{mp} + \delta_{lp}\delta_{mn}). \end{aligned} \tag{5·27}$$

Hence the coefficient in the asymptotic behaviour (5·26) may be written in terms of L_{lmnp}, and it is interesting to observe that the large-scale structure of the velocity covariance is determined by the same tensor coefficient as was the large-scale motion in the earlier work of Batchelor (1949) based on the assumption of convergent integral moments of $R_{ij}(\mathbf{r})$.

Returning, now, to the behaviour of the spectrum tensor $\Phi_{ij}(\mathbf{k})$ at small values of k, the principal simplification arising from the foregoing results appears to be due to the conditional (though not absolute) convergence of the integral L_{ijkl}. Thus the integral (5·2) exists at $\mathbf{k} = 0$, and, although it does not represent the second derivative of the spectrum at that point, the effect is to ensure that this derivative is uniformly bounded at all points (other than $\mathbf{k} = 0$) within a sphere of finite radius surrounding the origin of wave-number space. Thus (5·4) should be replaced by

$$\frac{\partial^2 \Phi_{ij}(\mathbf{k})}{\partial k_k \partial k_l} = O(1),$$

and we recover from (5·5) (with (5·7) and (5·8)) what is, perhaps, the most important single result concerning the large-scale motion, namely that

$$\Phi_{ij}(\mathbf{k}) = O(k^2) \tag{5·28}$$

for small values of k. In other words, the singularity in the leading term of the spectrum is concerned with the *direction* of $\mathbf{k}$ rather than its magnitude, and the task is, fundamentally, to ascertain this singular dependence of $\Phi_{ij}(\mathbf{k})$ on the direction of $\mathbf{k}$ from the directional properties of the asymptotic behaviour of $R_{ij}(\mathbf{r})$. However, since it is known that these simplifications arise from the peculiar behaviour of the vorticity covariance at large values of r, it is rather easier to derive the form of $\Phi_{ij}(\mathbf{k})$ from the spectrum of the vorticity fluctuations.

Hence, we define the vorticity spectrum $\Omega_{ij}(\mathbf{k})$ by the equation

$$\Omega_{ij}(\mathbf{k}) = \frac{1}{8\pi^3}\int V_{ij}(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r}, \tag{5·29}$$

and the Fourier transform of (5·9) then gives

$$\Omega_{ij}(\mathbf{k}) = (k^2\delta_{ij} - k_i k_j)\,\Phi_{kk}(\mathbf{k}) - k^2\Phi_{ji}(\mathbf{k}). \tag{5·30}$$

It follows from (5·5) that $\Omega_{ij}(\mathbf{k}) = o(k^3)$ for small values of k, so that $\Omega_{ij}(\mathbf{k})$ and its first three derivatives vanish at $\mathbf{k} = 0$. The essential point now is that the Taylor expansion of $\Omega_{ij}(\mathbf{k})$ may be carried one term further by virtue of the known behaviour $V_{ij}(\mathbf{r}) = O(r^{-8})$ for large values of r. Thus, the fourth integral moment of $V_{ij}(\mathbf{r})$, namely C_{ijlmnp}, is absolutely convergent, so that the fourth derivative of $\Omega_{ij}(\mathbf{k})$ is a continuous function of $\mathbf{k}$ at all values of $\mathbf{k}$, including $\mathbf{k} = 0$. Also the integrals

$$\int |\, r_l r_m r_n r_p r_q V_{ij}(\mathbf{r})\,|^2\, d\mathbf{r} \quad \text{and} \quad \int |\, r_l r_m r_n r_p r_q V_{ij}(\mathbf{r})\,|\, d\mathbf{r}$$

converge and diverge logarithmically, respectively, so that the fifth derivative of $\Omega_{ij}(\mathbf{k})$ is a continuous function of $\mathbf{k}$ for all non-zero values of $\mathbf{k}$, and is $O(\ln k)$ for small values of k. The behaviour of $\Omega_{ij}(\mathbf{k})$ at small values of k is therefore

$$\Omega_{ij}(\mathbf{k}) = \frac{1}{2\pi^2} C_{ijlmnp} k_l k_m k_n k_p + O(k^5 \ln k)$$

$$= \frac{1}{4\pi^2} \epsilon_{ilq} \epsilon_{jmr} C_{qrnp} k_l k_m k_n k_p + O(k^5 \ln k). \tag{5·31}$$

The behaviour of $\Phi_{ij}(\mathbf{k})$ is obtained by using the relation

$$\Omega_{kk}(\mathbf{k}) = k^2 \Phi_{kk}(\mathbf{k})$$

to write (5·30) in the form

$$\Phi_{ij}(\mathbf{k}) = \frac{1}{k^2}\left[\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\Omega_{kk}(\mathbf{k}) - \Omega_{ji}(\mathbf{k})\right].$$

Substitution from (5·31) then gives, after a little manipulation,

$$\Phi_{ij}(\mathbf{k}) = \frac{C_{lmnp}}{4\pi^2}\left(\delta_{il} - \frac{k_i k_l}{k^2}\right)\left(\delta_{jm} - \frac{k_j k_m}{k^2}\right) k_n k_p + O(k^3 \ln k), \tag{5·32}$$

for small values of k. Thus $\Phi_{ij}(\mathbf{k})$ is of order k^2 with a singularity at $\mathbf{k} = 0$ of the type already anticipated. The coefficient in (5·32) may also be expressed in terms of the tensor L_{lmnp}, but the equation is not then so simple. We should also note here that the spectrum function giving the distribution of energy with respect to wave-number magnitude k, namely

$$E(k) = \int \Phi_{ii}(\mathbf{k})\, \mathrm{d}S(\mathbf{k}),$$

where the integration is over the surface of a sphere of radius k with centre at the origin of $\mathbf{k}$-space, has the form

$$E(k) = Ck^4 + O(k^5 \ln k)$$

for small values of k, where

$$C = \frac{1}{2\pi} C_{lmnp}\left(\tfrac{1}{3}\delta_{lm}\delta_{np} - \tfrac{1}{15}\delta_{lmnp}\right).$$

Thus, although the integration over all directions removes the non-analytic nature of the leading term, this is not in general true of the function as a whole.

As a final check on the consistency of the analysis, we may derive the asymptotic behaviour (5·26) of $R_{ij}(\mathbf{r})$ directly from the result (5·32). Thus, writing

$$\Phi_{ij}(\mathbf{k}) = \left(\delta_{il} - \frac{k_i k_l}{k^2}\right)\left(\delta_{jm} - \frac{k_j k_m}{k^2}\right) \Psi_{lm}(\mathbf{k}),$$

where

$$\Psi_{lm}(\mathbf{k}) \sim \frac{1}{4\pi^2} C_{lmnp} k_n k_p \quad \text{as } k \to 0, \tag{5·33}$$

we have

$$R_{ij}(\mathbf{r}) = \int \Phi_{ij}(\mathbf{k})\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\, \mathrm{d}\mathbf{k}$$

$$= r^{-3}\left(\delta_{il}\nabla^2 - \frac{\partial^2}{\partial l_i \partial l_l}\right)\left(\delta_{jm}\nabla^2 - \frac{\partial^2}{\partial l_j \partial l_m}\right)\int \frac{\Psi_{lm}(\boldsymbol{\lambda}/r)}{\lambda^4}\, \mathrm{e}^{\mathrm{i}\boldsymbol{\lambda}\cdot\mathbf{l}}\, \mathrm{d}\boldsymbol{\lambda}, \tag{5·34}$$

where $\mathbf{l} = \mathbf{r}/r, \boldsymbol{\lambda} = r\mathbf{k}$, and the operator ∇^2 here stands for $\partial^2/\partial l_q \partial l_q$. The convergence of this integral is assured by (5·33), and when r is large we have

$$\int \frac{\Psi_{lm}(\boldsymbol{\lambda}/r)}{\lambda^4} \mathrm{e}^{\mathrm{i}\boldsymbol{\lambda}.\mathbf{l}} \mathrm{d}\boldsymbol{\lambda} \sim \frac{1}{4\pi^2} C_{lmnp} r^{-2} \int \frac{\lambda_n \lambda_p}{\lambda^4} \mathrm{e}^{\mathrm{i}\boldsymbol{\lambda}.\mathbf{l}} \mathrm{d}\boldsymbol{\lambda} = \frac{1}{4\pi^2} C_{lmnp} r^{-2} \frac{\partial^2}{\partial l_n \partial l_p} \int \frac{1-\mathrm{e}^{\mathrm{i}\boldsymbol{\lambda}.\mathbf{l}}}{\lambda^4} \mathrm{d}\boldsymbol{\lambda}$$

(where a term independent of $\mathbf{l}$ has been added to the integrand in order to make the integral convergent)

$$= \tfrac{1}{4} C_{lmnp} r^{-2} \frac{\partial^2 l}{\partial l_n \partial l_p}. \tag{5·35}$$

The asymptotic form (5·35) holds for all $\mathbf{l}$, so that it is permissible to find the asymptotic form of (5·34) by changing the order of the two operations of differentiation with respect to $\mathbf{l}$ and taking the limit as $r \to \infty$. Hence, as $r \to \infty$,

$$R_{ij}(\mathbf{r}) \sim \tfrac{1}{4} C_{lmnp} r^{-5} \left(\delta_{il} \nabla^2 - \frac{\partial^2}{\partial l_i \partial l_l}\right)\left(\delta_{jm} \nabla^2 - \frac{\partial^2}{\partial l_j \partial l_m}\right) \frac{\partial^2 l}{\partial l_n \partial l_p}$$

$$= \tfrac{1}{4} C_{lmnp} \left(\delta_{il} \nabla^2 - \frac{\partial^2}{\partial r_i \partial r_l}\right)\left(\delta_{jm} \nabla^2 - \frac{\partial^2}{\partial r_j \partial r_m}\right) \frac{\partial^2 r}{\partial r_n \partial r_p},$$

where ∇^2 stands for $\partial^2/\partial r_q \partial r_q$ again, in agreement with (5·26).

It will be noticed that the derivation of the asymptotic behaviour of $R_{ij}(\mathbf{r})$ via its Fourier transform is considerably shorter than the corresponding analysis in physical space. We have included the latter largely on account of its close analogy with the conventional treatment of the large-scale structure of irrotational velocity fields, which also owe their existence to the long-range effects of the pressure.

6. General dynamics of the large-scale motion

In this section we propose to examine the consequences of the Navier–Stokes equation, for the leading term in the velocity covariance $R_{ij}(\mathbf{r})$ (at large r) or, equivalently, for the leading term in the spectrum tensor $\Phi_{ij}(\mathbf{k})$ (at small k). The basic dynamical equation is

$$\frac{\partial R_{ij}(\mathbf{r})}{\partial t} = \frac{\partial}{\partial r_k}(\overline{u_i u_k u'_j} - \overline{u_i u'_k u'_j}) + \frac{\partial \overline{p u'_j}}{\partial r_i} - \frac{\partial \overline{p' u_i}}{\partial r_j} + 2\nu \nabla^2 R_{ij}(\mathbf{r}). \tag{6·1}$$

The explicit asymptotic form of $R_{ij}(\mathbf{r})$ (as $r \to \infty$) that is consistent with the basic hypothesis of §3 and with all the kinematical conditions, has been found in the previous section, and similar results for the terms on the right side of (6·1) are now required. As with the velocity covariance $R_{ij}(\mathbf{r})$, it is possible to obtain the asymptotic form of each of the terms in (6·1) at large r from the behaviour of their Fourier transforms at small k, and this is the procedure to be adopted here. (The same results can be obtained without the introduction of Fourier transforms, but the proofs are then longer.)

Kinematical relations for $\overline{u_i u_k u'_j}$

Consider first the two triple velocity product mean values in (6·1). We define

$$\Upsilon_{ikj}(\mathbf{k}) = \frac{1}{8\pi^3} \int \overline{u_i u_k u'_j}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}.\mathbf{r}} \mathrm{d}\mathbf{r}. \tag{6·2}$$

This integral exists, and is continuous, for all $\mathbf{k}$, since $\overline{u_i u_k u'_j}$ is continuous and is known to be of order r^{-4} when r is large. The incompressibility condition shows that

$$k_j \Upsilon_{ikj}(\mathbf{k}) = 0 \tag{6·3}$$

for all $\mathbf{k}$, one consequence of which is that

$$\Upsilon_{ikj}(0) = \frac{1}{8\pi^3}\int \overline{u_i u_k u_j'}\,\mathrm{d}\mathbf{r} = 0. \tag{6·4}$$

From (6·2) we have

$$\mathrm{i}\,\epsilon_{jlm}k_m\Upsilon_{ikj}(\mathbf{k}) = \Lambda_{ikl}(\mathbf{k}) = \frac{1}{8\pi^3}\int \overline{u_i u_k \omega_l'}\,\mathrm{e}^{-\mathrm{i}\mathbf{k}.\mathbf{r}}\,\mathrm{d}\mathbf{r}, \tag{6·5}$$

and, in view of (6·3),

$$\Upsilon_{ikj}(\mathbf{k}) = \mathrm{i}\,\epsilon_{jnl}\frac{k_n}{k^2}\Lambda_{ikl}(\mathbf{k}). \tag{6·6}$$

Now the mean value $\overline{u_i u_k \omega_l'}$ is of order r^{-6} when r is large, even though $\overline{u_i u_k u_l'}$ is of order r^{-4} (as established in §4), so that $\Lambda_{ikl}(\mathbf{k})$ and its first two derivatives exist, and are continuous, for all $\mathbf{k}$. Moreover, $\int |\overline{u_i u_k \omega_l'}\,r_n r_p r_q|^2\,\mathrm{d}\mathbf{r}$ converges, so that the third derivative of $\Lambda_{ikl}(\mathbf{k})$ exists for $k > 0$, although in general

$$\int_{r<R} \overline{u_i u_k \omega_l'}\,r_n r_p r_q\,\mathrm{d}\mathbf{r} = O(\ln R)$$

when R is large. Thus noting that the zero and first integral moments of $\overline{u_i u_k \omega_l'}$ vanish identically (after integration by parts), we can write the following Taylor series with remainder for $\Lambda_{ikl}(\mathbf{k})$:

$$\Lambda_{ikl}(\mathbf{k}) = -\frac{1}{2!}k_p k_q B_{iklpq} + \frac{\mathrm{i}}{3!}k_p k_q k_n \frac{1}{8\pi^3}\int \overline{u_i u_k \omega_l'}\,r_p r_q r_n\,\mathrm{e}^{-\mathrm{i}\alpha\mathbf{k}.\mathbf{r}}\,\mathrm{d}\mathbf{r}, \tag{6·7}$$

where $0 < |\alpha| \leqslant 1$, and

$$B_{iklpq} = \frac{1}{8\pi^3}\int \overline{u_i u_k \omega_l'}\,r_p r_q\,\mathrm{d}r.$$

The exponential in the integral in (6·7) is effectively equal to unity for values of r between zero and $O(k^{-1})$ so that, when k is small,

$$\Lambda_{ikl}(\mathbf{k}) = -\tfrac{1}{2}k_p k_q B_{iklpq} + O(k^3 \ln k), \tag{6·8}$$

and

$$\Upsilon_{ikj}(\mathbf{k}) = -\tfrac{1}{2}\mathrm{i}\,\epsilon_{jnl}\frac{k_n k_p k_q}{k^2}B_{iklpq} + O(k^2 \ln k). \tag{6·9}$$

From this expression for the form of the Fourier transform of $\overline{u_i u_k u_j'}$ at small values of k we now obtain the asymptotic form of $\overline{u_i u_k u_j'}$ as $r \to \infty$. We have, from (6·2) and (6·6),

$$\overline{u_i u_k u_j'} = \mathrm{i}\,\epsilon_{jnl}\int \frac{k_n}{k}\Lambda_{ikl}(\mathbf{k})\,\mathrm{e}^{\mathrm{i}\mathbf{k}.\mathbf{r}}\,\mathrm{d}\mathbf{k}.$$

$\Lambda_{ikl}(\mathbf{k})$ has been shown to be quadratic in $\mathbf{k}$ when k is small, and there is no loss of generality in putting

$$\Lambda_{ikl}(\mathbf{k}) = k_p k_q \Gamma_{iklpq}(\mathbf{k})$$

for all $\mathbf{k}$, where

$$\Gamma_{iklpq}(\mathbf{k}) \begin{cases} = -\tfrac{1}{2}B_{iklpq} & \text{when } k = 0, \\ = o(k^{-2}) & \text{as } k \to \infty. \end{cases} \tag{6·10}$$

Hence

$$\overline{u_i u_k u_j'} = -\epsilon_{jnl}r^{-4}\frac{\partial^3}{\partial l_n \partial l_p \partial l_q}\int \lambda^{-2}\Gamma_{iklpq}(\boldsymbol{\lambda}/r)\,\mathrm{e}^{\mathrm{i}\boldsymbol{\lambda}.\mathbf{l}}\,\mathrm{d}\boldsymbol{\lambda} \tag{6·11}$$

where $\mathbf{l} = \mathbf{r}/r$ and $\boldsymbol{\lambda} = r\mathbf{k}$, and, in view of (6·10),

$$\int \lambda^{-2}\Gamma_{iklpq}(\boldsymbol{\lambda}/r)\, e^{i\boldsymbol{\lambda}.\mathbf{l}}\, d\boldsymbol{\lambda} \sim -\tfrac{1}{2}B_{iklpq}\int \lambda^{-2}\, e^{i\boldsymbol{\lambda}.\mathbf{l}}\, d\boldsymbol{\lambda}$$
$$= -B_{iklpq}\pi^2/l,$$

as $r \to \infty$. This limit holds for all values of $\mathbf{l}$, so that when r is large (6·11) becomes

$$\overline{u_i u_k u'_j} \sim \pi^2 \epsilon_{jnl} B_{iklpq} r^{-4} \frac{\partial^3}{\partial l_n \partial l_p \partial l_q}\left(\frac{1}{l}\right)$$
$$= \pi^2 \epsilon_{jnl} B_{iklpq} \frac{\partial^3}{\partial r_n \partial r_p \partial r_q}\left(\frac{1}{r}\right). \tag{6·12}$$

The fact that $\overline{u_i u_k \omega'_l}$ is solenoidal in the index l and is of order r^{-6} when r is large imposes a constraint on the form of the coefficient B_{iklpq}. The general theorem of appendix B shows that in these circumstances we can write

$$B_{iklpq} = \epsilon_{lpm} B_{ikqm} + \epsilon_{lqm} B_{ikpm}, \tag{6·13}$$

where B_{ikqm} is symmetrical in i and k; then substitution in (6·9) and (6·12) gives

$$\Upsilon_{ikj}(\mathbf{k}) \sim ik_l\left(\delta_{jm} - \frac{k_j k_m}{k^2}\right) B_{iklm}, \tag{6·14}$$

when k is small, and

$$\overline{u_i u_k u'_j} \sim 2\pi^2 B_{iklm} \frac{\partial^3}{\partial r_j \partial r_l \partial r_m}\left(\frac{1}{r}\right), \tag{6·15}$$

when r is large.

The last step is to relate the coefficient B_{iklm} to an integral moment of $\overline{u_i u_k u'_j}$, instead of $\overline{u_i u_k \omega'_l}$. We multiply both sides of (6·13) by ϵ_{lpj}, and find

$$3B_{ikqj} - \delta_{qj} B_{ikll} = \epsilon_{lpj} B_{iklpq}$$
$$= \epsilon_{lpj} \frac{1}{8\pi^3} \int \overline{u_i u_k \omega'_l}\, r_p r_q\, d\mathbf{r}$$
$$= \frac{1}{8\pi^3} \int (\delta_{pm}\delta_{jn} - \delta_{pn}\delta_{jm}) \frac{\partial \overline{u_i u_k u'_n}}{\partial r_m} r_p r_q\, d\mathbf{r}. \tag{6·16}$$

Now, from (6·15) and with an argument similar to that leading to the convergence of the integral (5·19), we see that the first integral moment of $\overline{u_i u_k u'_j}$ converges, and we can define

$$T_{ikjl} = \frac{1}{4\pi} \int \overline{u_i u_k u'_j}\, r_l\, d\mathbf{r}. \tag{6·17}$$

Then (6·16) becomes, after integration by parts, and with the use of (6·15) in the surface integral

$$3B_{ikqj} - \delta_{jq} B_{ikll} = \frac{1}{8\pi^3}(\delta_{pm}\delta_{jn} - \delta_{pn}\delta_{jm})\left\{2\pi^3 B_{ikls}\int \frac{\partial^3}{\partial r_n \partial r_l \partial r_s}\left(\frac{1}{r}\right) r_p r_q r_m r\, d\Omega(\mathbf{r}) - 4\pi(\delta_{mp} T_{iknq} + \delta_{mq} T_{iknp})\right\}$$
$$= (\delta_{pm}\delta_{jn} - \delta_{pn}\delta_{jm})\{-\tfrac{1}{7}\delta_{nlspqm} + \tfrac{1}{5}(\delta_{nl}\delta_{spqm} + \delta_{ls}\delta_{npqm} + \delta_{sn}\delta_{lpqm})\} B_{ikls} - \frac{1}{2\pi^2}(3T_{ikjq} - \delta_{jq} T_{ikll})$$
$$= \tfrac{1}{5}(3B_{ikqj} + 3B_{ikjq} - 2\delta_{jq} B_{ikll}) - \frac{1}{2\pi^2}(3T_{ikjq} - \delta_{jq} T_{ikll}),$$

where the integral has been evaluated by the formulae given in appendix A. Then

$$12B_{ikqj}-3B_{ikjq}-3\delta_{jq}B_{ikll}=-\frac{5}{2\pi^2}(3T_{ikjq}-\delta_{jq}T_{ikll}), \tag{6·18}$$

and if this relation be combined with that obtained by interchanging the suffixes q and j, we obtain

$$3B_{ikqj}-\delta_{jq}B_{ikll}=-\frac{1}{2\pi^2}(4T_{ikjq}+T_{ikqj}-\tfrac{5}{3}\delta_{jq}T_{ikll}). \tag{6·19}$$

Substitution in (6·14) and (6·15) gives the final expressions for the asymptotic forms:

$$\Upsilon_{ikj}(\mathbf{k})\sim-\frac{\mathrm{i}}{6\pi^2}k_l\left(\delta_{jm}-\frac{k_jk_m}{k^2}\right)(4T_{ikml}+T_{iklm}) \tag{6·20}$$

when k is small, and

$$\overline{u_iu_ku_j'}\sim-\tfrac{5}{3}T_{iklm}\frac{\partial^3}{\partial r_j\,\partial r_l\,\partial r_m}\left(\frac{1}{r}\right) \tag{6·21}$$

when r is large.

Actually there is a further condition to be examined, although it has not been needed for the above results. The fact that $\overline{u_iu_ku_j'}$ is solenoidal in the index j imposes a constraint on the form of T_{iklm}. We begin with the identity

$$\int\frac{\partial\overline{u_iu_ku_j'}}{\partial r_j}r_lr_m\,\mathrm{d}\mathbf{r}=0, \tag{6·22}$$

and find, after integration by parts,

$$\begin{aligned}T_{iklm}+T_{ikml}&=\frac{1}{4\pi}\lim_{R\to\infty}\int_{r=R}\overline{u_iu_ku_j'}\,r_jr_lr_mr\,\mathrm{d}\Omega(\mathbf{r})\\&=-\tfrac{5}{3}T_{ikpq}\frac{1}{4\pi}\int\frac{\partial^3}{\partial r_j\,\partial r_p\,\partial r_q}\left(\frac{1}{r}\right)r_jr_lr_mr\,\mathrm{d}\Omega(\mathbf{r})\\&=T_{ikpq}(\delta_{lp}\delta_{mq}+\delta_{lq}\delta_{mp}-\tfrac{2}{3}\delta_{lm}\delta_{pq}),\end{aligned} \tag{6·23}$$

with further use of appendix A. Hence the condition reduces to

$$T_{ikll}=0. \tag{6·24}$$

It will be noticed that if for some special reason the mean value $\overline{u_iu_ku_j'}$ is instantaneously of smaller order than r^{-4} when r is large, the surface integral in (6·23) vanishes and T_{iklm} is anti-symmetrical in l and m. The asymptotic form (6·21) then vanishes identically, and (6·20) reduces to an analytic expression, as, in both cases, is to be expected. The incompressibility condition (6·24) is weaker than anti-symmetry of T_{iklm} in l and m and is valid more generally.

Kinematical relations for $\overline{pu_j'}$

Most of the corresponding results for the pressure-velocity covariance can be derived from the equation obtained by taking the divergence of (6·1), namely

$$\nabla^2\overline{pu_j'}=-\frac{\partial^2\overline{u_iu_ku_j'}}{\partial r_i\,\partial r_k}. \tag{6·25}$$

$\overline{pu_j'}$ is of order r^{-4} when r is large, so that the Fourier transform

$$\Pi_j(\mathbf{k})=\frac{1}{8\pi^3}\int\overline{pu_j'}\,\mathrm{e}^{-\mathrm{i}\mathbf{k}.\mathbf{r}}\,\mathrm{d}\mathbf{r}$$

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exists and is continuous for all $\mathbf{k}$. Then, from (6·25), (6·2) and (6·6),

$$\Pi_j(\mathbf{k}) = -\frac{k_i k_k}{k^2}\Upsilon_{ikj}(\mathbf{k}) = -\mathrm{i}\epsilon_{jnl}\frac{k_i k_k k_n}{k^4}\Lambda_{ikl}(\mathbf{k}), \tag{6·26}$$

whence, using (6·20), the asymptotic form of $\Pi_j(\mathbf{k})$ when k is small can be obtained at once.

The asymptotic form of $\overline{pu'_j}$ when r is large can now be found from the known behaviour of its Fourier transform near $\mathbf{k} = 0$ in much the same way as before. From (6·26) we have

$$\overline{pu'_j} = -\mathrm{i}\,\epsilon_{jnl}\int \frac{k_i k_k k_n}{k^4}\Lambda_{ikl}(\mathbf{k})\,\mathrm{e}^{\mathrm{i}\mathbf{k}.\mathbf{r}}\,\mathrm{d}\mathbf{k}$$

$$= \epsilon_{jnl} r^{-2}\frac{\partial^3}{\partial l_i \partial l_k \partial l_n}\int \lambda^{-4}\Lambda_{ikl}(\boldsymbol{\lambda}/r)\,\mathrm{e}^{\mathrm{i}\boldsymbol{\lambda}.\mathbf{l}}\,\mathrm{d}\boldsymbol{\lambda}, \tag{6·27}$$

where $\mathbf{l} = \mathbf{r}/r$ and $\boldsymbol{\lambda} = r\mathbf{k}$. The behaviour of $\Lambda_{ikl}(\mathbf{k})$ is described by (6·10), and when r is large

$$\int \lambda^{-4}\Lambda_{ikl}(\boldsymbol{\lambda}/r)\,\mathrm{e}^{\mathrm{i}\boldsymbol{\lambda}.\mathbf{l}}\,\mathrm{d}\boldsymbol{\lambda} \sim -\tfrac{1}{2}B_{iklpq} r^{-2}\int \frac{\lambda_p \lambda_q}{\lambda^4}\mathrm{e}^{\mathrm{i}\boldsymbol{\lambda}.\mathbf{l}}\,\mathrm{d}\boldsymbol{\lambda}$$

$$= -\tfrac{1}{2}\pi^2 B_{iklpq} r^{-2}\frac{\partial^2 l}{\partial l_p \partial l_q}$$

for all $\mathbf{l}$. Hence when r is large (6·27) becomes

$$\overline{pu'_j} \sim -\tfrac{1}{2}\pi^2\epsilon_{jnl}B_{iklpq}\frac{\partial^3}{\partial r_i \partial r_k \partial r_n}\left\{\frac{\partial^2 r}{\partial r_p \partial r_q}\right\}, \tag{6·28}$$

or, on making use of (6·13), and then of (6·19),

$$\overline{pu'_j} \sim \pi^2 B_{iklm}\left(\delta_{jm}\nabla^2 - \frac{\partial^2}{\partial r_j \partial r_m}\right)\frac{\partial^3 r}{\partial r_i \partial r_k \partial r_l}$$

$$= -\tfrac{1}{6}(4T_{ikml} + T_{iklm})\left(\delta_{jm}\nabla^2 - \frac{\partial^2}{\partial r_j \partial r_m}\right)\frac{\partial^3 r}{\partial r_i \partial r_k \partial r_l}. \tag{6·29}$$

The non-harmonic part of the expression (6·29) could of course have been obtained immediately from the differential equation (6·25) and the known asymptotic form of the right-hand side, but the determination of the harmonic part of the asymptotic form of $\overline{pu'_j}$ requires the use of some method like that above.

Since the asymptotic form (6·29) is a derivative of at least the second order, it follows that the integral $\int \overline{pu'_j}\, r_l\,\mathrm{d}\mathbf{r}$ converges. We shall show, for later use, that this integral is linearly related to the first integral moment of $\overline{u_i u_k u'_j}$. From equation (6·25) we have

$$\int \nabla^2 \overline{pu'_j}\, r_p r^2\,\mathrm{d}\mathbf{r} = -\int \frac{\partial^2 \overline{u_i u_k u'_j}}{\partial r_i \partial r_k} r_p r^2\,\mathrm{d}\mathbf{r}, \tag{6·30}$$

and two integrations by parts give

$$10\int \overline{pu'_j}\, r_p\,\mathrm{d}\mathbf{r} = \lim_{R\to\infty}\int_{r=R}\left\{-\frac{\partial \overline{pu'_j}}{\partial r_q} r_q r_p r^3 + 3\overline{pu'_j}\, r_p r^3 - \frac{\partial \overline{u_i u_k u'_j}}{\partial r_i} r_p r_k r^3 + \overline{u_i u_k u'_j}(\delta_{pk} r_i r^3 + 2r_i r_p r_k r)\right\}\mathrm{d}\Omega(\mathbf{r})$$

$$-\int \overline{u_i u_k u'_j}(2\delta_{pk} r_i + 2\delta_{ip} r_k + 2\delta_{ik} r_p)\,\mathrm{d}\mathbf{r}, \tag{6·31}$$

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and, on making use of (6·21) and (6·29) and of the rules for evaluating surface integrals described in appendix A,

$$= -\tfrac{10}{3}T_{iklm}\int \frac{\partial^3}{\partial r_j\,\partial r_l\,\partial r_m}\left(\frac{1}{r}\right) r_i r_k r_p r\,\mathrm{d}\Omega(\mathbf{r}) - 4\pi(4T_{lpjl}+2T_{lljp})$$

$$= -\tfrac{10}{3}T_{iklm}\,4\pi\{-\tfrac{1}{7}\delta_{ikpjlm}+\tfrac{1}{5}(\delta_{jl}\delta_{mikp}+\delta_{lm}\delta_{jikp}+\delta_{mj}\delta_{likp})\} - 4\pi(4T_{lpjl}+2T_{lljp}),$$

so that, making some use of (6·24),

$$\frac{1}{4\pi}\int \overline{pu_j'}\,r_p\,\mathrm{d}\mathbf{r} = \tfrac{1}{105}(10\delta_{jp}T_{kllk}+10T_{ljpl}-46T_{lpjl}+10T_{jllp}-4T_{pllj}-23T_{lljp}-2T_{llpj}). \quad (6\cdot 32)$$

The vector $\overline{pu_j'}$ is solenoidal, so that a relation like (6·22) holds. The consequent condition on $\int \overline{pu_j'}\,r_p\,\mathrm{d}\mathbf{r}$, corresponding to (6·24), is readily found to be

$$\frac{1}{4\pi}\int \overline{pu_j'}\,r_j\,\mathrm{d}\mathbf{r} = 0; \quad (6\cdot 33)$$

however, this result is already contained in (6·32).

The asymptotic form of the dynamical equation

As a result of the investigations in this and the preceding section, we are now in a position to write down the asymptotic form of equation (6·1) as $r\to\infty$, or of its Fourier transform as $k\to 0$. On substituting (5·35), (6·21) and (6·29) in (6·1) we find, with some rearrangement of the terms,

$$\left\{\frac{1}{4}\frac{\mathrm{d}C_{pqmn}}{\mathrm{d}t}-\tfrac{1}{6}(4T_{mpqn}+4T_{mqpn}+T_{pmnq}+T_{qmnp})\right\}\left(\delta_{ip}\nabla^2-\frac{\partial^2}{\partial r_i\,\partial r_p}\right)\left(\delta_{jq}\nabla^2-\frac{\partial^2}{\partial r_j\,\partial r_q}\right)\frac{\partial^2 r}{\partial r_m\,\partial r_n}=0, \quad (6\cdot 34)$$

where, it will be recalled, C_{pqmn} is related to the second integral moment of $\overline{u_i u_j'}$ by (5·27) and T_{pqmn} is the first integral moment of $\overline{u_p u_q u_m'}$ (see 6·17). The corresponding equation obtained by comparing terms of the second degree in k in the Fourier transform of (6·1) is

$$\frac{1}{\pi^2}\left\{\frac{1}{4}\frac{\mathrm{d}C_{pqmn}}{\mathrm{d}t}-\tfrac{1}{6}(4T_{mpqn}+4T_{mqpn}+T_{pmnq}+T_{qmnp})\right\}\left(\delta_{ip}-\frac{k_i k_p}{k^2}\right)\left(\delta_{jq}-\frac{k_j k_q}{k^2}\right)k_m k_n = 0. \quad (6\cdot 35)$$

These are the (equivalent) equations that express the effect of inertia and pressure forces on the large-scale structure of the turbulence. They are of so complex a form as to be unlikely to yield predictions of the kind that could be observed in a wind tunnel. It also seems unlikely that there are any simple dynamical results about the large-scale structure that can be put in the place of the old erroneous 'result' that the asymptotic form of the energy spectrum at small k is independent of time.

It is not possible to equate the expression within curly brackets in (6·34) and (6·35) to zero, because there are several tensor forms (e.g. δ_{mp}) which vanish when contracted with the remainder of the left side of (6·34) or (6·35). Consequently, to obtain an explicit expression for $\mathrm{d}C_{pqmn}/\mathrm{d}t$ or $\mathrm{d}L_{pqmn}/\mathrm{d}t$ we must adopt a different procedure. Equation (6·1) must be multiplied by $r_p r_q$ and then integrated over all values of $\mathbf{r}$, the resulting volume and surface integrals on the right-hand side being evaluated with the aid of the results obtained in this section, namely (6·21), (6·29) and (6·32). In this way $\mathrm{d}C_{pqmn}/\mathrm{d}t$ and $\mathrm{d}L_{pqmn}/\mathrm{d}t$ are found to be (rather lengthy) linear functions of T_{ijmn}, and in general are non-zero. From the point of view of dynamics of the large-eddy structure, as represented by the behaviour of $R_{ij}(\mathbf{r})$ at large r or of $\Phi_{ij}(\mathbf{k})$ at small k, it is of course not the non-zero character of the rate of change of L_{pqmn} that is relevant, but rather that of C_{pqmn} contracted with the functions of $\mathbf{r}$ or $\mathbf{k}$ in (6·34) or (6·25). In other words, (6·34) and (6·35) alone contain the essential information about the rate of change of the asymptotic forms of the correlation and spectrum functions.

7. The final period of decay

Having established the general form of the spectrum tensor for small wave-numbers, we can proceed to determine the spectrum tensor for arbitrary values of $\mathbf{k}$ in the final period of decay, using the customary method (see Batchelor 1953). The assumption underlying a discussion of the 'final period of decay' is that when the time of decay is sufficiently large, inertia forces become small compared with viscous forces, and the decay of the turbulence is described by the linear equation

$$\frac{\partial \mathbf{u}}{\partial t} = \nu \nabla^2 \mathbf{u}. \tag{7·1}$$

This assumption was regarded as intuitively plausible when it was first made, and that is still our view of it. The changes made necessary by the developments reported herein lie in the deduction of the consequences of equation (7·1).

It follows from (7·1) that

$$\frac{\partial R_{ij}(\mathbf{r},t)}{\partial t} = 2\nu \nabla^2 R_{ij}(\mathbf{r},t), \tag{7·2}$$

or equivalently,

$$\frac{\partial \Phi_{ij}(\mathbf{k},t)}{\partial t} = -2\nu k^2 \Phi_{ij}(\mathbf{k},t), \tag{7·3}$$

of which the solution is

$$\Phi_{ij}(\mathbf{k},t) = \Phi_{ij}(\mathbf{k},t_0)\, \mathrm{e}^{-2\nu k^2(t-t_0)}, \tag{7·4}$$

where t_0 is a virtual origin for the final period of decay. As $t-t_0 \to \infty$, the exponential factor becomes very small for all except small values of k, and the asymptotic form (as $t-t_0 \to \infty$) is obtained by replacing $\Phi_{ij}(\mathbf{k},t_0)$ in (7·4) by its leading term when k is small. This leading term has been found to be of order k^2, with a form given by (5·32) (the remainder of $\Phi_{ij}(\mathbf{k})$ being of order $k^3 \ln k$). Hence, as $t-t_0 \to \infty$,

$$\Phi_{ij}(\mathbf{k},t) \sim \frac{1}{4\pi^2} C_{pqmn}\left(\delta_{ip}-\frac{k_i k_p}{k^2}\right)\left(\delta_{jq}-\frac{k_j k_q}{k^2}\right) k_m k_n\, \mathrm{e}^{-2\nu k^2(t-t_0)}. \tag{7·5}$$

The tensor coefficient C_{pqmn} is expressed in terms of the fourth integral moment of the vorticity covariance by (5·25) and in terms of the second integral moment of $R_{ij}(\mathbf{r})$ by (5·27). During the final period of decay, when (7·1) is valid, C_{pqmn} is constant (as may be seen by comparing the asymptotic forms of the two sides of (7·4) as $\mathbf{k} \to 0$), but at earlier stages of the decay C_{pqmn} is not constant, as indicated by equation (6·35). Thus in the above solution for $\Phi_{ij}(\mathbf{k},t)$, C_{pqmn} must be regarded as a constant coefficient whose value is determined, in a complicated way, by the characteristics of the mechanism producing the turbulence and by the subsequent decay; the value of C_{pqmn} in the final period is beyond the scope of theoretical prediction at the moment.

Corresponding to (7·5), we have, as $t-t_0 \to \infty$,

$$R_{ij}(\mathbf{r},t) \sim \int \frac{1}{4\pi^2} C_{pqmn}\left(\delta_{ip}-\frac{k_i k_p}{k^2}\right)\left(\delta_{jq}-\frac{k_j k_q}{k^2}\right) k_m k_n \, \mathrm{e}^{\mathrm{i}\mathbf{k}.\mathbf{r}-2\nu k^2(t-t_0)} \, \mathrm{d}\mathbf{k}$$

$$= \frac{1}{4\pi^2} C_{pqmn}\left(\delta_{ip}\nabla^2-\frac{\partial^2}{\partial r_i \partial r_p}\right)\left(\delta_{jq}\nabla^2-\frac{\partial^2}{\partial r_j \partial r_q}\right)\frac{\partial^2}{\partial r_m \partial r_n}\int\left(\frac{1-\cos\mathbf{k}.\mathbf{r}}{k^4}\right)\mathrm{e}^{-2\nu k^2(t-t_0)}\,\mathrm{d}\mathbf{k}, \tag{7·6}$$

where a term independent of $\mathbf{r}$ has been added to the integrand in order to make the integral in (7·6) convergent. The integral can be evaluated by elementary methods, whence it is found that, as $t-t_0 \to \infty$,

$$R_{ij}(\mathbf{r},t) \sim \left[\frac{\nu(t-t_0)}{2\pi}\right]^{\frac{1}{2}} C_{pqmn}\left(\delta_{ip}\nabla^2-\frac{\partial^2}{\partial r_i \partial r_p}\right)\left(\delta_{jq}\nabla^2-\frac{\partial^2}{\partial r_j \partial r_q}\right)\frac{\partial^2}{\partial r_m \partial r_n}\left[\mathrm{e}^{-y^2}+\pi^{\frac{1}{2}}\left(y+\frac{1}{2y}\right)\operatorname{erf} y\right], \tag{7·7}$$

where $y = r/[8\nu(t-t_0)]^{\frac{1}{2}}$ and $\operatorname{erf} y = 2/\sqrt{\pi}\int_0^y \mathrm{e}^{-x^2}\,\mathrm{d}x$.

The expression (7·7) is of the form

$$[\nu(t-t_0)]^{-\frac{5}{2}} \times \text{function of } \mathbf{r}/[8\nu(t-t_0)]^{\frac{1}{2}}.$$

so that the mean-square of each component of the velocity diminishes as $[\nu(t-t_0)]^{-\frac{5}{2}}$ in the final period of decay, as was predicted by the earlier investigations. This prediction is of course a direct consequence of the spectrum tensor being of the second degree in $\mathbf{k}$ when k is small. The explicit expression for the energy tensor $\overline{u_i u_j}$ in the final period is most readily found by returning to (7·5):

$$\overline{u_i u_j} = R_{ij}(0,t) = \frac{1}{4\pi^2} C_{pqmn}\int\left(\delta_{ip}-\frac{k_i k_p}{k^2}\right)\left(\delta_{jq}-\frac{k_j k_q}{k^2}\right) k_m k_n \, \mathrm{e}^{-2\nu k^2(t-t_0)}\,\mathrm{d}\mathbf{k}$$

$$= \frac{3}{8\pi^{\frac{1}{2}}}[\nu(t-t_0)]^{-\frac{5}{2}} C_{pqmn}\left[\tfrac{1}{3}\delta_{ip}\delta_{jq}\delta_{mn}-\tfrac{1}{15}(\delta_{ip}\delta_{jqmn}+\delta_{jq}\delta_{ipmn})+\tfrac{1}{105}\delta_{ijpqmn}\right]. \tag{7·8}$$

The expression (7·7) retains the power-law form at large values of r that has been shown to be characteristic of the correlation tensor at earlier stages of the decay. For the expression within square brackets on which the derivatives operate can be written as

$$\pi^{\frac{1}{2}}\left(y+\frac{1}{2y}\right)+\left[\mathrm{e}^{-y^2}+\pi^{\frac{1}{2}}\left(y+\frac{1}{2y}\right)(\operatorname{erf} y-1)\right],$$

of which the second term is exponentially small when y is large and the first term gives rise to the correct asymptotic (as $r \to \infty$) form already given in (5·26). It will be noted that the vorticity covariance $\overline{\omega_i \omega_j'}$ is exponentially small at large values of r in the final period of decay.

As a further check on the accuracy of the work, one may note that if for some special reason $R_{ij}(\mathbf{r})$ is of smaller order than r^{-5} when r is large* (as would be the case if the spectrum tensor were analytic in $\mathbf{k}$) (7·7) reduces to the simple form obtained in earlier papers. For in these circumstances we have

$$L_{ijmn} = (\epsilon_{imp}\epsilon_{jnq}+\epsilon_{inp}\epsilon_{jmq})M_{pq}, \tag{7·9}$$

* This assumption is kinematically permissible at any instant, and in the final period of decay it would remain true at all subsequent instants since the effects of inertia and pressure forces are then negligible.

according to the formulae in appendix B (M_{pq} being an undetermined tensor, symmetrical in p and q); then, from (5·27),

$$C_{pqmn} = -L_{pqmn} + \text{terms not contributing to (7·7)},$$

so that (7·7) becomes

$$R_{ij}(\mathbf{r},t) \sim -\left[\frac{\nu(t-t_0)}{2\pi}\right]^{\frac{1}{2}} L_{ijmn} \nabla^4 \frac{\partial^2}{\partial r_m \partial r_n}\left[e^{-y^2} + \pi^{\frac{1}{2}} \operatorname{erf} y \left(y + \frac{1}{2y}\right)\right]$$

$$= \frac{2}{\pi^{\frac{1}{2}}}[8\nu(t-t_0)]^{-\frac{3}{2}} L_{ijmn} \frac{\partial^2}{\partial r_m \partial r_n} e^{-r^2/8\nu(t-t_0)}, \tag{7·10}$$

which is identical with the expression previously thought to be correct (compare with equations (5.4.6) and (5.3.16) of Batchelor, 1953). Equation (7·10) (together with (7·9)) has the specially simple feature that, as $t-t_0 \to \infty$,

$$\frac{r_i r_j}{r^2} R_{ij}(\mathbf{r},t) \sim -\frac{4}{\pi^{\frac{1}{2}}}[8\nu(t-t_0)]^{-\frac{5}{2}} L_{ijmn} e^{-r^2/8\nu(t-t_0)}$$

$$= \left[\frac{r_i r_j}{r^2} R_{ij}(\mathbf{r},t)\right]_{r=0} e^{-r^2/8\nu(t-t_0)}; \tag{7·11}$$

that is, the longitudinal correlation coefficient has a Gaussian form which is independent of the direction of the separation $\mathbf{r}$. The more general expression (7·7) does not seem to have any such simple property.

It seems, then, that the only useful prediction about the final period of decay that can be made on the basis of the present work is that the mean-square of each component of the velocity diminishes as $(t-t_0)^{-\frac{5}{2}}$. No simple results about the forms of the spectrum and correlation tensors have emerged. These forms have been found explicitly, but the presence of an unknown numerical tensor of fourth order makes difficult any comparison with measurements. It is not possible to relate the energy tensor (7·8) to conditions at the initial instant of generation of the homogeneous turbulence, so that, in particular, there is no proof that anisotropy of the mechanism generating the turbulence requires anisotropy of the turbulence in the final period. Unless we have overlooked some deduction, conditions in the final period are a good deal less simple than has hitherto been supposed.

The measurements of turbulence at an advanced stage of decay that have been made (Batchelor & Townsend 1948, Batchelor & Stewart 1950) are already known to agree with the prediction that the energy decays as $(t-t_0)^{-\frac{5}{2}}$; this provides welcome support for our analysis, and for the basic hypothesis of § 3, although with such a complicated situation it is difficult to be sure that we have not reached the right result for the wrong reason. The measurements have also established, beyond question, that the turbulence behind a grid of bars is markedly anisotropic in the final period,* the mean-square of the downstream component of velocity being about 50 % greater than that of either cross-stream component. A further experimental result is that the longitudinal correlation coefficient has a form that is represented approximately by $\exp[-r^2/8\nu(t-t_0)]$ (see figure 4 of Batchelor & Townsend 1948). These latter two results were believed to be explained quite well by the earlier

* A related observation (Batchelor & Stewart 1950) is that for a grid consisting of one set of parallel bars, the lateral correlation coefficient for large values of the separation r is not isotropic during the initial period of decay.

theoretical work, and it is disconcerting that the present more extensive analysis cannot do as well. A suggestion that we put forward tentatively to account for these observations is that they may not be general and may be peculiar to cases of homogeneous turbulence generated at relatively low Reynolds number. When the initial Reynolds number of the turbulence is relatively small, the duration of the period of decay during which inertia forces are not negligible is small, and the final period begins at a comparatively early stage (which is the experimental reason for using low Reynolds numbers—the final period of decay would not otherwise occur within the available length of the working-section of a wind tunnel). According to our basic hypothesis of § 3, the power-law asymptotic forms of the correlation functions develop wholly as a result of the action of inertia and pressure forces, and our suggestion is that when the duration of action of inertia forces is small the turbulence in the final period of decay will not be very different from that which would be predicted with the assumption of an analytic spectrum.

An alternative possibility is that all the effects of the non-analytic form of the spectrum are small, at both low and high Reynolds numbers. We have not been able to obtain any results that support or exclude this possibility. An examination of the point involves the very difficult question of numerical values of the tensor integral moments (such as T_{ikjl}) occurring in the analysis.

8. The special case of isotropic turbulence

In this section we shall record the form taken by the principal results when the turbulence is assumed to be completely isotropic, making use of the usual rules concerning the forms of isotropic tensors.

The main result that has been obtained for homogeneous turbulence generally is the expression (5·26) for the asymptotic form of $R_{ij}(\mathbf{r})$ as $r \to \infty$. Now when the turbulence is wholly isotropic, the numerical tensor C_{lmnp}, which is related to the fourth integral moment of the vorticity covariance, and which is symmetrical in l and m and in n and p, is necessarily of the form

$$C_{lmnp} = A\delta_{lm}\delta_{np} + B(\delta_{ln}\delta_{mp} + \delta_{lp}\delta_{mn}), \tag{8·1}$$

where A and B are undetermined scalars; the asymptotic form (5·26) is then identically zero. It was shown that the remainder was in general $O(r^{-6})$, so that in isotropic turbulence $R_{ij}(\mathbf{r})$ is no larger than $O(r^{-6})$ when r is large. This result is related to the fact that the only second-order isotropic tensor whose double curl (one with respect to i, the other with respect to j) vanishes—as is required of the term of order r^{-5} in $R_{ij}(\mathbf{r})$, since it makes no contribution to the vorticity covariance—is

$$\text{const.} \times \frac{\partial^2}{\partial r_i \partial r_j}\left(\frac{1}{r}\right),$$

which is not of the right order of magnitude. The longitudinal correlation coefficient of $f(r)$, which determines $R_{ij}(\mathbf{r})$ completely, will likewise be no larger than $O(r^{-6})$ when r is large.

(The above argument does not, of course, given any indication of whether the term of order r^{-6} remains non-zero in isotropic turbulence. We have been unable to find any term in the expressions for $\partial R_{ij}/\partial t$, $\partial^2 R_{ij}/\partial t^2$, $\partial^3 R_{ij}/\partial t^3$, etc. at the initial instant that remains non-zero when the condition of isotropy is imposed; however, neither have we been able to prove that such a non-zero term *cannot* arise. Thus the order of magnitude of $f(r)$ at large r—and, relatedly, of $\frac{d}{dr}[r^4 k(r)]$—is still an open question, and the possibility that these two quantities are exponentially small cannot be excluded.

It should be emphasized that the point is of theoretical interest only, since the large-scale structure of fields of turbulence generated behind grids is known to be markedly anisotropic. Thus, notwithstanding the remarks of the present section, it is the general results for homogeneous turbulence that must be compared with measurements of the turbulence behind a grid. Measured values of the longitudinal correlation coefficient behind grids would still be expected to be asymptotically proportional to r^{-5}.)

The fact that $R_{ij}(\mathbf{r})$ is now of smaller order than r^{-5} when r is large ensures that the second integral moment of $R_{ij}(\mathbf{r})$ is absolutely convergent, and that the second derivative of $\Phi_{ij}(\mathbf{k})$ exists and is continuous for all $\mathbf{k}$, including $\mathbf{k} = 0$. The asymptotic form of $\Phi_{ij}(\mathbf{k})$ when $\mathbf{k}$ is small contains no singularity, and substitution of (8·1) in (5·33) gives

$$\Phi_{ij}(\mathbf{k}) = \frac{A}{4\pi^2}(\delta_{ij}k^2 - k_i k_j) + O(k^3 \ln k) \tag{8·2}$$

when k is small. Moreover, since

$$L_{ijmn} = \frac{1}{4\pi}\int r_m r_n R_{ij}(\mathbf{r})\,d\mathbf{r}$$

is absolutely convergent and $R_{ij}(\mathbf{r})$ is solenoidal in i and j, we have (see Appendix B)

$$L_{ijmn} + L_{imnj} + L_{injm} = 0.$$

This, together with the isotropy, requires L_{ijmn} to be of the form

$$L_{ijmn} = \tfrac{1}{3}L(-\delta_{ij}\delta_{mn} + \tfrac{1}{2}\delta_{im}\delta_{jn} + \tfrac{1}{2}\delta_{in}\delta_{jm}), \tag{8·3}$$

in which we have put
$$L = \overline{u^2}\int_0^\infty r^4 f(r)\,dr,$$

where $\overline{u^2}$ represents the mean square of any component of the velocity. Then the scalars A and L can be related with the aid of

$$\left[\frac{\partial^2 \Phi_{ij}(\mathbf{k})}{\partial k_m \partial k_n}\right]_{k=0} = -\frac{1}{2\pi^2}L_{ijmn},$$

which requires
$$A = \tfrac{1}{3}L. \tag{8·4}$$

Turning now to the dynamical results, let us consider first the relatively simple matter of the final period of decay. We have just seen that, when the turbulence is isotropic, L_{ijmn} is subject to the same kinematical conditions as it would be if all integral moments converged. The asymptotic form of the spectrum tensor at small values of k depends only on L_{ijmn}, so we conclude that in the final period of decay of isotropic turbulence the results previously published are correct. That is, we have, as $t - t_0 \to \infty$,

$$\overline{u^2} \sim \frac{L}{48(2\pi)^{\frac{1}{2}}[\nu(t-t_0)]^{\frac{5}{2}}}, \quad f(r) \sim e^{-r^2/8\nu(t-t_0)} \tag{8·5}$$

(As already remarked in §7, some measurements (Batchelor & Townsend 1948) of the longitudinal correlation coefficient in a wind tunnel are in reasonably good agreement with the Gaussian function given in (8·5), but this cannot be taken as confirmation of the prediction (8·5) since the turbulence in the wind tunnel was definitely not isotropic.)

During the initial period of decay, the non-linear terms of the dynamical equation must be taken into account, and we must consider the results obtained in § 6. The pressure-velocity mean value $\overline{pu_j'}$ is identically zero in isotropic turbulence, and makes no contribution to the asymptotic form of the dynamical equation (6·1). The asymptotic form of the first term on the right-hand side of (6·1) depends on the tensor T_{ijkl} representing the first integral moment of $\overline{u_i u_k u_j'}$ (see (6·17)), and in isotropic turbulence we must have

$$T_{ikjl} = T\delta_{ik}\delta_{jl} + S(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}). \tag{8·6}$$

The general condition (6·24) then shows that the two undetermined scalars T and S are related by

$$3T + 2S = 0,$$

and in place of (6·21) we have

$$\overline{u_i u_k u_j'} \sim 5T \frac{\partial^3}{\partial r_i \partial r_k \partial r_j}\left(\frac{1}{r}\right), \tag{8·7}$$

when r is large. It will be observed that the asymptotic form of $\overline{u_i u_k u_j'}$ is now irrotational in all three indices, which is wholly a consequence of isotropy and the fact that the asymptotic form is of order r^{-4} (as may be seen by substituting $k(r) = \text{const.}/r^4$ in the known relation between $\overline{u_i u_k u_j'}$ and the single scalar function $k(r)$). Thus the significance of the irrotationality with respect to the j-index, which was so clear in the general case—being associated with the growth of irrotational velocity fields as a result of the action of pressure forces—is here rather obscured.

The divergence of the asymptotic form (8·7) with respect to the k-index vanishes identically so that in isotropic turbulence no terms of the dynamical equation (6·1) are of order r^{-5} when r is large. Correspondingly, none of the leading terms (of order k^2) in the dynamical equation for the spectrum tensor is non-analytic, and the relation (6·35) between the leading terms in this latter dynamical equation reduces to

$$\frac{1}{\pi^2}\left[\frac{1}{12}\frac{dL}{dt} + \tfrac{5}{2}T\right](k^2\delta_{ij} - k_i k_j) = 0,$$

from which we have

$$\frac{dL}{dt} = -30T. \tag{8·8}$$

L and T are scalars defining the second integral moment of $\overline{u_i u_j'}$ and the first integral moment of $\overline{u_i u_k u_j'}$ respectively, and (8·8) can be rewritten in terms of the usual correlation coefficients $f(r)$ and $k(r)$. We find

$$T = -\tfrac{1}{30}(\overline{u^2})^{\frac{3}{2}} \int_0^\infty \left(r^4 \frac{\partial k(r)}{\partial r} + 4r^3 k(r)\right) dr,$$

so that

$$\frac{d}{dt}\left[\overline{u^2}\int_0^\infty r^4 f(r)\, dr\right] = (\overline{u^2})^{\frac{3}{2}} \lim_{r\to\infty} r^4 k(r). \tag{8·9}$$

This relation can of course be obtained directly by taking the fourth integral moment of the dynamical equation for the scalar function $f(r)$, viz.

$$\frac{\partial \overline{u^2} f(r)}{\partial t} = (\overline{u^2})^{\frac{3}{2}}\left(\frac{\partial k(r)}{\partial r} + \frac{4k(r)}{r}\right) + 2\nu\overline{u^2}\left(\frac{\partial^2 f(r)}{\partial r^2} + \frac{4}{r}\frac{\partial f(r)}{\partial r}\right).$$

The practice, prior to the work of Proudman & Reid, has been to assume $\lim_{r\to\infty} r^4 k(r)$ to be zero, which leads to the conclusion that $\overline{u^2}\int_0^\infty r^4 f(r)\,\mathrm{d}r$ is invariant throughout the history of the turbulence. Proudman & Reid's inference, from their work on turbulence with zero fourth-order cumulants, that $\lim_{r\to\infty} r^4 k(r)$ is not zero and that L is not invariant, is now confirmed. Thus the parameter L occurring in the expression for the kinetic energy in the final period of decay (see (8·5)) is not a quantity that is determined at the instant of formation of the field of turbulence; the history of the decay also affects the value of L.

Proudman & Reid (1954) established that, for a field of turbulence in which fourth-order cumulants of the velocity are zero, $\mathrm{d}T/\mathrm{d}t < 0$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(\overline{u^2})^{\frac{3}{2}} \lim_{r\to\infty} r^4 k(r)\right] > 0, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\overline{u^2}\int_0^\infty r^4 f(r)\,\mathrm{d}r\right] > 0. \tag{8·10}$$

Measurements of $k(r)$ and $f(r)$ suggest that they are everywhere negative and positive respectively (although the data for large values of r is incomplete), and the interpretation of these inequalities may be simply that the variation of the quantities within square brackets is dominated by the decrease of $\overline{u^2}$ as t increases. It seems that the quantity $\overline{u^2}\int_0^\infty r^4 f(r)\,\mathrm{d}r$ decreases (at a diminishing rate) during the stages of the decay in which inertia forces are not negligible, and is constant during the final period of the decay.

Appendix A. Evaluation of some isotropic surface integrals

In the analysis of §§5 and 6, there occur a number of isotropic surface integrals, all of which are particular cases of the two integrals

$$S_{ijklmn} = \int r_i r_j \frac{\partial^4 r}{\partial r_k \partial r_l \partial r_m \partial r_n}\, r\,\mathrm{d}\Omega(\mathbf{r}) \tag{A 1}$$

and

$$T_{ijklmn} = \int r_i r_j r_k \frac{\partial^3}{\partial r_l \partial r_m \partial r_n}\left(\frac{1}{r}\right) r\,\mathrm{d}\Omega(\mathbf{r}), \tag{A 2}$$

where $\mathrm{d}\Omega(\mathbf{r})$ is the solid angle subtended at the origin by a surface element at the point $\mathbf{r}$, and the integration is over the surface of a sphere. The purpose of this appendix is to evaluate these two integrals.

In view of the symmetry properties of the integrals, it is convenient to introduce special symbols for completely symmetric isotropic tensors of order up to six. The second-order isotropic tensor is, of course, the unit tensor δ_{ij}. For the fourth- and sixth-order cases, we define

$$\delta_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \tag{A 3}$$

and

$$\delta_{ijklmn} = \delta_{ij}\delta_{klmn} + \delta_{ik}\delta_{jlmn} + \delta_{il}\delta_{jkmn} + \delta_{im}\delta_{jkln} + \delta_{in}\delta_{jklm}, \tag{A 4}$$

which are clearly symmetric in all their indices.

Considering first the integral (A 1), the symmetry in i, j and in k, l, m, n enables the integral to be written in the form

$$S_{ijklmn} = S\delta_{ijklmn} + S'\delta_{ij}\delta_{klmn}, \tag{A 5}$$

where S and S' are scalar numbers. Then, since r is a bi-harmonic function, the contracted form S_{ijkkmm} vanishes, and (A 5) gives

$$0 = 35S\delta_{ij} + 15S'\delta_{ij}. \tag{A 6}$$

A further relation between S and S' follows from a consideration of the contracted form S_{ijijmn}. For,

$$S_{ijijmn} = \int(-2)\,(-1)\,\frac{\partial^2 r}{\partial r_m \partial r_n}\,r\,\mathrm{d}\Omega(\mathbf{r})$$

$$= 2\int\left(\delta_{mn} - \frac{r_m r_n}{r^2}\right)\mathrm{d}\Omega(\mathbf{r}) = \frac{16\pi}{3}\delta_{mn},$$

where the first step follows from the rules for differentiating homogeneous functions of degree -2 and -1. Hence, from (A 5),

$$\frac{16\pi}{3}\delta_{mn} = 35S\delta_{mn} + 5S'\delta_{mn},$$

which, combined with (A 6), gives

$$S_{ijklmn} = \frac{8\pi}{5}\left[\tfrac{1}{7}\delta_{ijklmn} - \tfrac{1}{3}\delta_{ij}\,\delta_{klmn}\right] \tag{A 7}$$

The symmetry of the second integral (A 2) again enables the integral to be expressed in terms of two scalars; this time in the form

$$T_{ijklmn} = T\delta_{ijklmn} + T'(\delta_{lm}\,\delta_{nijk} + \delta_{ln}\,\delta_{mijk} + \delta_{mn}\,\delta_{lijk}). \tag{A 8}$$

The values of the scalars T and T' are then most easily obtained from the result that, when the indices k and l are contracted, the integrals (A 1) and (A 2) are proportional. Thus,

$$T_{ijkkmn} = -3\int r_i r_j \frac{\partial^2}{\partial r_m \partial r_n}\left(\frac{1}{r}\right) r\,\mathrm{d}\Omega(\mathbf{r})$$

$$= -\tfrac{3}{2}S_{ijkkmn},$$

so that, from (A 7) and (A 8),

$$(7T + 2T')\,\delta_{ijmn} + 5T'\delta_{ij}\,\delta_{mn} = -\frac{12\pi}{5}\left[\delta_{ijmn} - \tfrac{5}{3}\delta_{ij}\,\delta_{mn}\right].$$

This last equation determines both T and T', and on substituting these values in (A 8) we obtain

$$T_{ijklmn} = 4\pi\left[-\tfrac{1}{7}\delta_{ijklmn} + \tfrac{1}{5}(\delta_{lm}\,\delta_{nijk} + \delta_{ln}\,\delta_{mijk} + \delta_{mn}\,\delta_{lijk})\right].$$

Appendix B. The form of integral moments of solenoidal tensors

The kinematical analysis in §§ 5 and 6 involves a number of integral moments of correlation tensors which satisfy a solenoidal condition. In this appendix, we derive the general explicit solution for the tensorial form of such integral moments.

The first problem is to obtain the form of the Nth-order tensor

$$F_{jkl\ldots p} = \int r_k r_{l\ldots}(N-1 \text{ factors}) \ldots r_p A_j(\mathbf{r})\,\mathrm{d}\mathbf{r}, \tag{B 1}$$

where $A_j(\mathbf{r})$ is a solenoidal vector, and the integral is supposed to be absolutely convergent. This tensor therefore satisfies the conditions

$$F_{jkl\ldots p} \text{ is symmetric in } k, l, \ldots, p, \tag{B 2}$$

and

$$F_{jkl\ldots p} + F_{kl\ldots pj} + F_{l\ldots pjk} + \ldots (N \text{ terms}) = 0, \tag{B 3}$$

where (B3) follows by the same argument as that which led to (5·23). Now a form of $F_{ikl...p}$ that satisfies these two conditions is

$$F_{jkl...p} = \epsilon_{jkq} G_{ql...p} + \epsilon_{jlq} G_{q...pk} + \ldots (N-1 \text{ terms}), \tag{B4}$$

where $G_{ql...p}$ is symmetric in $l, \ldots, p$. The tensor $G_{ql...p}$ is also arbitrary to the extent of an additive tensor of the form

$$\delta_{ql} H_{m...p} + \delta_{qm} H_{n...pl} + \ldots (N-2 \text{ terms}), \tag{B5}$$

where $H_{m...p}$ is symmetric in $m, \ldots, p$, since the contribution of (B5) to the right side of (B4) vanishes identically.* The proof that (B4) represents the general solution of the conditions (B2) and (B3) then rests upon the demonstration that the number of independent components in a general tensor satisfying these conditions is the same as that in the symmetric tensor $G_{ql...p}$, allowing for the arbitrariness represented by (B5).

Let D_n denote the number of combinations of n things chosen from three varieties, with repetitions allowed; so that

$$D_n = \tfrac{1}{2}(n+1)(n+2).$$

The number of independent components in an Nth-order tensor with the symmetry property (B2) is

$$3D_{N-1} = \tfrac{3}{2}N(N+1),$$

and the condition (B3) imposes a further D_N relation between these components. Hence the number of independent components in the general solution for $F_{jkl...p}$ is

$$\tfrac{3}{2}N(N+1) - \tfrac{1}{2}(N+1)(N+2) = N^2 - 1. \tag{B6}$$

Similarly, the tensors $G_{ql...p}$ and $H_{m...p}$ have $3D_{N-2}$ and D_{N-3} independent components, respectively, so that the number of independent components in the solution (B4) is

$$\tfrac{3}{2}(N-1)N - \tfrac{1}{2}(N-2)(N-1) = N^2 - 1,$$

in agreement with (B6). The solution (B4) is therefore general, and, with a suitable choice of $H_{m...p}$, we may put

$$G_{qqm...p} = 0. \tag{B7}$$

The validity of the above results is clearly not affected significantly by the replacement of the vector $A_j(\mathbf{r})$ by a tensor of any order that is solenoidal in the j-index; the only necessary change being the purely formal addition of extra indices to the tensor $G_{ql...p}$. If, however, there are further symmetry conditions involving the indices of the tensor that replaces $A_j(\mathbf{r})$, then a further reduction of the result (B4) follows. The only such case occurring in the analysis of the present paper is that in which $A_j(\mathbf{r})$ is replaced by a second-order solenoidal tensor $A_{ij}(\mathbf{r})$ satisfying the condition

$$A_{ij}(\mathbf{r}) = A_{ji}(-\mathbf{r}) \tag{B8}$$

(e.g. the fourth integral moment of the vorticity covariance), and this is discussed below.

Writing

$$F_{ijkl...p} = \int r_k r_l \ldots (N-1 \text{ factors}) \ldots r_p A_{ij}(\mathbf{r})\, d\mathbf{r}, \tag{B9}$$

where N is odd, the condition (B8) gives

$$F_{ijkl...p} = F_{jikl...p}, \tag{B10}$$

* The result corresponds to the fact that a solenoidal vector may be written as the curl of a vector potential which is arbitrary to the extent of an additive gradient.

and hence, from (B4),

$$F_{ijkl\ldots p} = \epsilon_{jkq} G_{iql\ldots p} + \epsilon_{jlq} G_{iq\ldots pk} + \ldots (N-1 \text{ terms})$$
$$= \epsilon_{ikq} G_{jql\ldots p} + \epsilon_{ilq} G_{jq\ldots pk} + \ldots (N-1 \text{ terms}). \quad \text{(B 11)}$$

Multiplying (B 11) by ϵ_{jkr}, and putting

$$G_{iqqm\ldots p} = 0$$

(which is permissible by (B 7)), we get

$$2G_{irlm\ldots p} + (N-2)\, G_{irlm\ldots p} = G_{irlm\ldots p} - \delta_{ir} G_{sslm\ldots p}$$
$$+ (N-2)\, G_{irlm\ldots p} + \sum_{\text{perm. } l, m, \ldots, p} [-G_{lrim\ldots p} + \delta_{lr} G_{ssim\ldots p} - \delta_{ir} G_{sslm\ldots p}].$$

Contraction of the indices i and r in this last equation yields the result

$$G_{iilm\ldots p} = 0, \quad \text{(B 12)}$$

so that the equation becomes
$$\sum_{\text{perm. } i, l, \ldots, p} G_{irl\ldots p} = 0. \quad \text{(B 13)}$$

Thus the tensor $G_{irl\ldots p}$, introduced in (B 11), also satisfies a condition of the type (B 3), and the general solution of (B 13) may be written in the form

$$G_{irlm\ldots p} = \epsilon_{ils} K_{rsm\ldots p} + \epsilon_{ims} K_{rs\ldots pl} + \ldots (N-2 \text{ terms}), \quad \text{(B 14)}$$

where the $(N-1)$th-order tensor $K_{rsm\ldots p}$ is symmetric in $m, \ldots, p$. The result (B 12) yields the further property that $K_{rsm\ldots p}$ is symmetric in r, s; to which we may also add the condition

$$K_{rssn\ldots p} = 0,$$

in a manner analogous to (B 7). In terms of $K_{rsm\ldots p}$, the explicit solution for $F_{ijklm\ldots p}$ is, from (B 11) and (B 14)

$$F_{ijklm\ldots p} = \sum_{\text{perm. } k, l, m, \ldots, p} \; \sum_{\text{perm. } l, m, \ldots, p} \epsilon_{ikr} \epsilon_{jls} K_{rsm\ldots p} \, [(N-1)\,(N-2) \text{ terms}]$$

Finally, it should be noted that the absolute convergence of the integral moment is essential for the validity of the results of this appendix. For moments that are only conditionally convergent, like L_{ijlm}, the zero on the right-hand side of the solenoidal condition (B 3) must be replaced by a surface integral whose value is determined by the local behaviour of the integrand at large values of r. In such cases, a 'particular' solution must be added to the 'complementary' solutions derived above.

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